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Inclusion–exclusion principle for belief functions

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A B S T R A C T

The inclusion–exclusion principle is a well-known property in probability theory, and is instrumental in some computational problems such as the evaluation of system reliability or the calculation of the probability of a Boolean formula in diagnosis. However, in the setting of uncertainty theories more general than probability theory, this principle no longer holds in general. It is therefore useful to know for which families of events it continues to hold. This paper investigates this question in the setting of belief functions. After exhibiting original sufficient and necessary conditions for the principle to hold, we illustrate its use on the uncertainty analysis of Boolean and non-Boolean systems in reliability.

1. Introduction

Probability theory is the most well-known approach to model uncertainty. However, even when the existence of a single probability measure is assumed, it often happens that its distribution is only partially known. This is particularly the case in the presence of severe uncertainty (few samples, imprecise or unreliable data, etc.) or when subjective beliefs are elicited (e.g., from experts). Some authors use a selection principle that brings us back to a precise distribution (e.g., maximum entropy [23]), but other ones [28,26,16] have argued that in some situations involving imprecision or incompleteness, uncertainty cannot be modelled faithfully by a single probability measure. The same authors have advocated the need for frameworks accommodating imprecision, their efforts resulting in different frameworks such as possibility theory [16], belief functions [26], imprecise probabilities [28], info-gap theory [4], etc. that are formally connected [29,17]. Regardless of interpretive issues, the formal setting of belief functions offers a good compromise between expressiveness and calculability, as it is more general than probability theory, yet in many cases remains more tractable than imprecise probability approaches.

Nevertheless using belief functions is often more computationally demanding than using probabilities. Indeed, its higher level of generality prevents the use of some properties, valid in probability theory, that help simplify calculations. This is the case, for instance, for the well-known and useful inclusion–exclusion principle (also known as Sylvester–Poincaré equality).

Given a space $\mathscr{X}$, a probability measure P over this space and any collection $\mathscr{A}_n = \{A_1, \ldots, A_n | A_i \subseteq \mathscr{X}\}$ of measurable subsets of $\mathscr{X}$, the inclusion–exclusion principle states that

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$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\mathscr{I} \subseteq \mathscr{A}_n} (-1)^{|\mathscr{I}|+1} P\left(\bigcap_{A \in \mathscr{I}} A\right) \tag{1}$$

where $|\mathscr{I}|$ is the cardinality of $\mathscr{I}$. This equality allows us to easily compute the probability of $\bigcup_{i=1}^{n} A_i$, when the events A_i are stochastically independent, or when their intersections are disjoint. This principle has been applied to numerous problems, including the evaluation of the reliability of complex systems. It does not hold for belief functions, and only an inequality remains. However, it is useful to investigate whether or not an equality can be restored for specific families $\mathscr{A}_n$ of events, in particular the ones encountered in applications to diagnosis and reliability. The main contribution of this paper is to give a positive answer to this question and to provide conditions characterising the families of events for which the inclusion–exclusion principle still holds in the belief function setting.

This paper is organised as follows. First, Section 2 provides sufficient and necessary conditions under which the inclusion–exclusion principle holds for belief functions in general spaces; it is explained why the question may be more difficult for the conjugate plausibility functions. Section 3 then studies how the results apply to the practically interesting case where events A_i and focal elements are Cartesian products in a multidimensional space. Section 4 investigates the particular case of binary spaces, and considers the calculation of the degree of belief and plausibility of a Boolean formula expressed in Disjunctive Normal Form (DNF). Section 5 then shows that specific events described by means of monotone functions over a Cartesian product of totally ordered discrete spaces meet the conditions for the inclusion–exclusion principle to hold. Section 6 is devoted to illustrative applications of the preceding results to the field of reliability analysis (both for the binary and non-binary cases), in which the use of belief functions is natural and the need for efficient computation schemes is an important issue. Finally, Section 7 compares our results with those obtained when assuming stochastic independence between ill-known probabilities, displaying those cases for which these results coincide and those for which they disagree.

This work extends the results concerning the computation of uncertainty bounds within the belief function framework previously presented in [22,1]. In particular, we provide full proofs as well as additional examples. We also discuss the application of the inclusion/exclusion principle to plausibilities, as well as a comparison of our approach with other types of independence notions proposed for imprecise probabilities (two issues not tackled in [22,1]).

2. General additivity conditions for belief functions

After introducing some notations and the basics of belief functions (Section 2.1), we explore in Section 2.2 general conditions for families of subsets for which the inclusion–exclusion principle holds for belief functions. We then look more closely at the specific case where the focal elements of belief functions are Cartesian products of subsets. Readers not interested in technical details and familiar with belief functions may directly move to Section 3.

2.1. Setting

A mass distribution [26] defined on a (finite) space $\mathscr{X}$ is a mapping $m : 2^{\mathscr{X}} \to [0,1]$ from the power set of $\mathscr{X}$ to the unit interval such that $m(\emptyset) = 0$ and $\sum_{E \subseteq \mathscr{X}} m(E) = 1$. A set E that receives a strictly positive mass is called a *focal element*, and the set of focal elements of m is denoted by $\mathscr{F}_m$. The mass function m can be seen as a probability distribution over sets, in this sense it captures both probabilities and sets: any probability p can be modelled by a mass m such that $m(\{x\}) = p(x)$ and any set E can be modelled by the mass $m(E) = 1$. In the setting of belief functions, a focal element is understood as a piece of incomplete information of the form $x \in E$ for some parameter x of interest. Then $m(E)$ can be understood as the probability that all that is known about x is that $x \in E$; in other words, $m(E)$ is a probability mass that should be divided over elements of E but is not, due to a lack of information.

From the mapping m are usually defined two set-functions, the belief and the plausibility functions, respectively defined for any $A \subseteq \mathscr{X}$ as

$$Bel(A) = \sum_{E \subseteq A} m(E), \tag{2}$$

$$Pl(A) = \sum_{E \cap A \neq \emptyset} m(E) = 1 - Bel\left(A^c\right), \tag{3}$$

with A^c the complement of A. They satisfy $Bel(A) \leq Pl(A)$. The belief function, which sums all masses of subsets that **imply** A, measures how much event A is certain, while the plausibility function, which sums all masses of subsets **consistent** with A, measures how much the event A is possible. Within the so-called theory of evidence [26], belief and plausibility functions are interpreted as confidence degrees about the event A, and are not necessarily related to probabilities. However, the mass distribution m can also be interpreted as the random set corresponding to an imprecisely observed random variable [12], and the measures Bel and Pl can be interpreted as describing a set of probabilities, that is, we can associate to them a set $\mathscr{P}(Bel)$ such that

$$\mathscr{P}(Bel) = \left\{P \,\middle|\, \forall A, Bel(A) \leq P(A) \leq Pl(A)\right\}$$