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Compositional models in valuation-based systems

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ABSTRACT

Compositional models were initially described for discrete probability theory, and later extended for possibility theory and for belief functions in Dempster–Shafer (D–S) theory of evidence. Valuation-based system (VBS) is an unifying theoretical framework generalizing some of the well known and frequently used uncertainty calculi. This generalization enables us to not only highlight the most important theoretical properties necessary for efficient inference (analogous to Bayesian inference in the framework of Bayesian network), but also to design efficient computational procedures. Some of the specific calculi covered by VBS are probability theory, a version of possibility theory where combination is the product t-norm, Spohn's epistemic belief theory, and D–S belief function theory. In this paper, we describe compositional models in the general framework of VBS using the semantics of no-double counting, which is central to the VBS framework. Also, we show that conditioning can be expressed using the composition operator. We define a special case of compositional models called decomposable models, again in the VBS framework, and demonstrate that for the class of decomposable compositional models, conditioning can be done using local computation. As all results are obtained for the VBS framework, they hold in all calculi that fit in the VBS framework. For the D–S theory of belief functions, the compositional model defined here differs from the one studied by Jiroušek, Vejnarová, and Daniel. The latter model can also be described in the VBS framework, but with a combination operator that is different from Dempster's rule of combination. For the version of possibility theory in which combination is the product t-norm, the compositional model defined here reduces to the one studied by Vejnarová.

1. Introduction

The framework of valuation-based systems (VBS) was introduced in [28,32,34] . The main idea behind VBS is to capture the common features of various uncertainty calculi and other domains such as optimization, decision-making theories, database systems, and solving systems of equations. Briefly, knowledge about a set of variables is represented by a set of functions called valuations. Each valuation is associated with a subset of variables. There are two operators called combination and marginalization. Combination allows us to aggregate knowledge, and marginalization allows us to coarsen knowledge to a smaller set of variables. The combination of all valuations, called the joint valuation, represents the joint knowledge of all variables. Making inferences can be described as finding marginals of the joint valuation for variables of interest. The VBS framework can be used to describe various uncertainty theories such as probability theory, a version of possibility theory where combination is the product t-norm [43], Spohn's epistemic belief theory [37,30], and Dempster-Shafer (D-S) belief function theory [26]. It can also be used to describe, e.g., propositional logic [29], solving systems of equations [25],

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optimization using dynamic programming [2,31], Bayesian decision-making by maximizing expected utility [33], relational database theory [42], and other domains [24].

Besides the marginalization and combination operators in the VBS framework, we define an additional operator called removal. Removal is an inverse of combination, and is useful for defining conditionals in the VBS framework. Conditionals are useful in characterizing conditional independence relations. All of these operators are required to satisfy some basic properties described as axioms. These axioms enable us to make inferences using local computation, using architectures such as the Shenoy-Shafer architecture [35] that uses only the combination and marginalization operators, and the Lauritzen–Spiegelhalter architecture [20] that uses the combination, marginalization, and removal operators. The main focus of VBS is to enable local computation of marginals of the joint valuation.

The VBS framework has been expanded, and studied further in greater mathematical depth. Shafer [27] provides an axiomatic treatment of conditionals called continuers, which are defined without explicit reference to a removal operator. Lauritzen and Jensen [20] describe an alternative axiomatization of the removal operator. Kohlas [17] studies VBS using abstract algebra, and also studies a class of VBS (called information algebras) where the valuations are idempotent. Kohlas and Wilson [18] link VBS to the algebraic theory of semirings. Finally, Pouly and Kohlas [24] describe local computation in VBS in great detail, including different architectures, and normalization, and provide many examples of domains that fit in the VBS framework.

In a Bayesian network model, one usually starts with a specification of the joint probability distribution that is factorized into conditionals for each variable given a subset of variables. The joint probability distribution is then obtained as the combination of all the conditionals, i.e., a fundamental assumption of a Bayesian network model is that there is no double-counting of knowledge in combining all conditionals to form the joint distribution. In a compositional model, one starts from a different starting point. One starts with a set of marginal probability distributions, where each marginal distribution is for some subset of variables. We cannot combine the marginal distributions as this would lead to double counting of knowledge (for those variables that are in the intersections of subsets of variables for which we have marginals). This is why we use the composition operator because it allows us to aggregate knowledge in the marginal distributions without double counting of knowledge. We assume that each variable is included in some subset for which we have marginals. This goal can also be reached by the iterative proportional fitting procedure (IPFP) [5]. The IPFP solution is obtained by an iterative procedure of high computational complexity, where at each step I-projections of multidimensional probability distributions are computed. To substantially decrease the computational complexity of this process, Perez proposed an approximate solution [23] based on his idea of dependence structure simplifications. The approximation consists in the fact that not all marginals from the given set are taken into consideration.

Another popular method for representing complex models from sets of marginal distributions and a dependence structure is the method based on Sklar's copulas [36]. But while it is computationally difficult to apply copulas to problems of more than 10 variables, IPFP (especially when using its decomposable representation) can be applied to problems of several tens of variables. Perez's approximation and compositional models can be applied to problems with hundreds of variables.

The goal of this paper is to describe compositional models in the general framework of VBS. The composition operator, which is the central operator of compositional models, was first introduced in probability theory to compare Csiszár's I-projections [4] and Perez's dependence structure simplifications [23], and to make it easier to understand the differences between these two concepts. Soon after, the composition operator was used to introduce compositional models, as an alternative to Bayesian networks, in the framework of discrete probability theory [10,11]. These models were later extended in [40] for possibility theory, and in [16] for belief functions in the D–S belief function theory.

In this paper, we use the VBS framework [34] to extend compositional models to all uncertainty calculi captured by the VBS framework, which includes calculi such as probability theory, a version of possibility theory with the product t-norm, Spohn's epistemic belief theory, and D–S belief function theory. We define a composition operator for valuations, and notice that conditional valuations can be described using the composition operator. Next, we define a class of compositional models called decomposable models, and for this class of models, we describe how conditioning can be done using local computation.

As the VBS framework includes the D–S theory of belief function, we have implicitly defined a compositional model for the D–S theory. We compare this compositional model with the one defined in [16] for belief functions. The two models are different. The compositional model described in [16] can be described in the VBS framework, but with a combination operator that is different from Dempster's rule of combination. Thus, the compositional model described in [16] is not for the D–S belief function theory that necessarily entails Dempster's rule of combination, but for an alternative belief function theory with the new rule of combination.

For the D–S belief function theory, if we remove a basic probability assignment (BPA) from another BPA, the resulting function may not be a BPA as the probability masses can be negative. This is true even if the BPA being removed is a marginal of the BPA it is being removed from. In this paper, we define a class of belief function models, called graphical belief, such that if we remove a BPA from another, the result is always a BPA.

We compare the VBS compositional model with the one described in [40] for possibility theory. The VBS framework captures only the version of possibility theory where the combination rule is the product t-norm. For this version of possibility theory, the two compositional models coincide. For the other versions of possibility theory, the combination rules (non-product t-norms) do not satisfy the axioms that the VBS operators are required to satisfy. Thus the applications of the local computation algorithms, such as the Shenoy-Shafer architecture [35] or the Lauritzen-Spiegelhalter architecture [20], are