International Journal of Approximate Reasoning 54 (2013) 1066–1086

Contents lists available at SciVerse ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Bilattices for deductions in multi-valued logic

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ARTICLE INFO

Article history:
Received 15 August 2012
Received in revised form 3 April 2013
Accepted 3 April 2013
Available online 11 April 2013

Keywords:
Multi-valued logic
Bilattice
Fuzzy logic
Heap paradox
Logic programming
Fuzzy control

ABSTRACT

In this exploratory paper we propose a framework for the deduction apparatus of multi-valued logics based on the idea that a deduction apparatus has to be a tool to manage information on truth values and not directly truth values of the formulas. This is obtained by embedding the algebraic structure $\boldsymbol{V}$ defined by the set of truth values into a bilattice $\boldsymbol{B}$. The intended interpretation is that the elements of $\boldsymbol{B}$ are pieces of information on the elements of $\boldsymbol{V}$. The resulting formalisms are particularized in the framework of fuzzy logic programming. Since we see fuzzy control as a chapter of multi-valued logic programming, this suggests a new and powerful approach to fuzzy control based on positive and negative conditions.

1. Introduction

In J. Pavelka's approach to fuzzy logic the truth values play two different roles (see [23]). The usual role, shared with the tradition of multi-valued logic, is semantic in nature and it is devoted to a truth-functional valuation of the formulas. A different role, connected with the inferential processes, is to represent information. Indeed, denote by F the set of formulas and by $\boldsymbol{V}$ the algebraic structure whose domain V is the set of truth values of the considered logic. Then the available information is represented by a fuzzy set of formulas $v : F \to V$. Given $\alpha \in F$, the intended meaning of $v(\alpha)$ is that the truth value of α is greater or equal to $v(\alpha)$. This means that $v(\alpha)$ is not interpreted as a truth value but as a piece of information on the unknown truth-value of α.

In this paper we propose some formalisms for fuzzy logic in which the truth dimension and the information dimension are clearly differentiated. In fact, we assume that while the semantics is fully represented by the algebraic structure $\boldsymbol{V}$, to define the deduction apparatus we have to extend $\boldsymbol{V}$ into a bilattice $\boldsymbol{B}$. This could give an answer to the important question denounced by D. Dubois in [5], i.e. the existing confusion between truth-values and information states. Notice that the first proposal to connect fuzzy logic with bilattice theory was done probably by E. Turunen, M. Öztürk, A. Tsoukiás in [25] in connection with paraconsistent logic. Our approach is different since we attempt to distinguish the role of the bilattice, related with information and inference, from the role of the valuation structure, semantics in nature. Instead, in [24] a bilattice (enriched with further operations) is directly considered as the valuation structure.

In the paper we test the proposed formalisms on a "negative" and "implication-free" version of the heap paradox. Moreover, we focus our attention in fuzzy logic programming (see P. Vojtas [26]) and in fuzzy control (see [13,14]). Notice that this is only an exploratory paper in which we sketch some ideas and formalisms. All the results are obvious adaptations of well known techniques.

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http://dx.doi.org/10.1016/j.ijar.2013.04.004

2. Preliminaries

The following definition was proposed by M. Ginsberg in [16] and M. Fitting in [7–9].

Definition 2.1. A (*complete*) *bilattice* is a structure $\boldsymbol{B} = (B, \leq_t, \leq_k, False, True, \perp, \top)$ such that both the reducts $(B, \leq_t, False, True)$ and $(B, \leq_k, \perp, \top)$ are complete lattices. A *negation*in $\boldsymbol{B}$ is a map $\sim : B \to B$ such that:

$$\sim\sim(x) = x; \quad x \leq_t y \Rightarrow \sim y \leq_t \sim x; \quad x \leq_k y \Rightarrow \sim x \leq_k \sim y.$$

The relations $\leq_t$ and $\leq_k$ are called *truth order* and *knowledge order*, respectively. We say that b is *negative* in the case $b \leq_k False$, *positive*in the case $b \leq_k True$. It is immediate that $\sim False = True$, $\sim True = False$, $\sim\perp = \perp$, $\sim \top = \top$ and that b is positive if and only if $\sim b$ is negative. Indexes t and k are used to distinguish the same notions defined with respect to $\leq_t$ and $\leq_k$. For example, we denote by $\wedge_t$ and $\vee_t$ the meet and join with respect to $\leq_t$ and by $\wedge_k$ and $\vee_k$ the same operations with respect to $\leq_k$. The meaning of the operators Sup_t, Inf_t, Sup_k and Inf_k is evident. A complete bilattice $\boldsymbol{B}$ is *infinitely distributive* provided that each of the connectives $\wedge_t, \vee_t, \wedge_k$ and $\vee_k$ is infinitely distributive with respect to any other..

Given a nonempty set S we denote by $\boldsymbol{B}^S$ the power of $\boldsymbol{B}$ with index set S. We look at an element in $\boldsymbol{B}^S$ as at a generalized characteristic function and, in accordance with fuzzy set theory, we adopt a nomenclature set-theoretical in nature. So, we call *B-subset* of S an element $s : S \to B$ of $\boldsymbol{B}^S$ and *n-ary B-relation*a B-subset of a Cartesian product of n sets. Also, we denote by $\subseteq_t$ and $\subseteq_k$ the order relations in $\boldsymbol{B}^S$ corresponding to $\leq_t$ and $\leq_k$ and by $\cup_t, \cup_k, \cap_t, \cap_k$ and- the operations in $\boldsymbol{B}^S$ corresponding to $\vee_t, \vee_k, \wedge_t, \wedge_k$ and $\sim$, respectively. Finally, we indicate by $s_\perp$ and $s_\top$ the B-subsets defined by setting $s_\perp(x) = \perp$ and $s_\top(x) = \top$, for every $x \in S$. The following proposition gives the main examples of bilattices (see [9]).

Proposition 2.2. *Let $\boldsymbol{L} = (L, \leq, 0, 1)$ be a complete lattice and denote by $\boldsymbol{B}(\boldsymbol{L})$ the structure $(L \times L, \leq_t, \leq_k, \sim, False, True, \perp, \top)$ obtained by setting for every $(x, x'), (y, y') \in L \times L$*

- $(x, x') \leq_t (y, y') \Leftrightarrow x \leq y$ *and* $y' \leq x'$*;*
- $(x, x') \leq_k (y, y') \Leftrightarrow x \leq y$ *and* $x' \leq y'$*;*
- $False = (0, 1)$, $True = (1, 0)$*;*
- $\perp = (0, 0)$, $\top = (1, 1)$*;*
- $\sim (x, x') = (x', x)$.

Then $\boldsymbol{B}(\boldsymbol{L})$ is a complete bilattice with negation called square bilattice associated with $\boldsymbol{L}$.

In the case $L = [0, 1]$ the relation $\leq_t$ is also called Pareto ordering (see for example [2]). One can think that the components x, x' of a pair (x, x') summarize the evidence for and the evidence against an assertion. The set of positive elements is $\{(x, 0) : x \in L\}$, the set of negative elements is $\{(0, y) : y \in L\}$. In a square bilattice we have that for every $(x, x'), (y, y') \in L \times L$:

- $(x, x') \wedge_t (y, y') = (x \wedge y, x' \vee y')$ *and* $(x, x') \vee_t (y, y') = (x \vee y, x' \wedge y')$*,*
- $(x, x') \wedge_k (y, y') = (x \wedge y, x' \wedge y')$ *and* $(x, x') \vee_k (y, y') = (x \vee y, x' \vee y')$*,*

where $\wedge$ and $\vee$ are the meet and the join in $\boldsymbol{L}$. Moreover, if C is a subset of $L \times L$, then

$$Sup_k(C) = (Sup(Pr_1(C)), Sup(Pr_2(C)), Inf_k(C) = (Inf(Pr_1(C)), Inf(Pr_2(C))$$

where Pr_1 and Pr_2 are the projection operators.

Further bilattices are defined by considering the intervals in $\boldsymbol{L}$ (see [24]). We recall that, given a, b in $\boldsymbol{L}$, the closed interval $[a, b]$ is defined by setting $[a, b] = \{x \in L : a \leq x \leq b\}$. In particular, since $[1, 0] = \emptyset$, the empty set is considered as a closed interval.

Proposition 2.3. *Let $\boldsymbol{L} = (L, \leq, 0, 1)$ be a complete lattice with an involution $\sim$ and denote by $Int(L)$ the set of closed intervals in $\boldsymbol{L}$. Define the structure $\boldsymbol{Int}(\boldsymbol{L}) = (Int(L), \leq_t, \leq_k, \sim, \{0\}, \{1\}, L, \emptyset)$ such that*

- *$\leq_k$ is the dual of the inclusion relation,*
- *for every $[a, b], [c, d]$ in $Int(L) - \{\emptyset\}$, $[a, b] \leq_t [c, d]$ provided that $a \leq c$ and $b \leq d$,*
- *$\{0\} \leq_t \emptyset \leq_t \{1\}$ and $\emptyset$ is not t-comparable with any other interval,*
- *$\sim [a, b] = [\sim b, \sim a]$; $\sim \emptyset = \emptyset$.*

Then $\boldsymbol{Int}(\boldsymbol{L})$ is a complete bilattice with a negation we call interval bilattice defined by $\boldsymbol{L}$.

Observe that the k-meet of two intervals $[a, b]$ and $[a', b']$ is the interval $[a \wedge a', b \vee b']$ while the k-join is the usual intersection $[a, b] \cap [a', b']$. The singletons $\{\{a\} : a \in L\}$ are the k-maximal intervals, the positive and negative elements are intervals as [a,1] and [0,b], respectively.

In this paper we refer mainly to square bilattices. This is not restrictive since, given an involution $\sim$ in $\boldsymbol{L}$, the map $h : Int(L) - \{\emptyset\} \to L \times L$ obtained by associating any nonempty interval $[a, b]$ with the pair $h([a, b]) = (a, \sim b)$ is an embedding of the structure $\boldsymbol{Int}(\boldsymbol{L}) - \{\emptyset\}$ into $\boldsymbol{B}(\boldsymbol{L})$.