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Sets with type-2 operations

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A R T I C L E I N F O

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A B S T R A C T

The algebra of truth values of type-2 fuzzy sets consists of all mappings of the unit interval to itself, with type-2 operations that are convolutions of ordinary max and min operations. This paper is concerned with a special subalgebra of this truth value algebra, namely the set of nonzero functions with values in the two-element set {0, 1}. This algebra can be identified with the set of all non-empty subsets of the unit interval, but the operations are not the usual union and intersection. We give simplified descriptions of the operations and derive the basic algebraic properties of this algebra, including the identification of its automorphism group. We also discuss some subalgebras and homomorphisms between them and look briefly at t-norms on this algebra of sets.

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1. Introduction

Type-2 fuzzy sets were introduced by Zadeh [21], extending the notion of ordinary fuzzy sets. There is now a rather extensive literature on the subject, discussing both theoretical and practical aspects. See, for example, [4–6, 9,10, 13–20]. The papers [19,20] give a mathematical treatment of the algebra of truth values for type-2 fuzzy sets and some of its subalgebras and its automorphisms, and we will refer to many of the results in those two papers.

The elements of the algebra **M** of truth values of type-2 fuzzy sets are the mappings of the unit interval [0, 1] into itself. Its operations are certain convolutions of operators on the unit interval, which we will describe later. This algebra has a number of interesting subalgebras, both from a theoretical and practical viewpoint. The algebra **M** and its subalgebras are examples of De Morgan bisemilattices, which have applications in multi-valued simulations of digital circuits and in hazard detection [1–3,7,11].

The truth value algebra of type-1, or ordinary fuzzy sets, is isomorphic to a subalgebra of **M**, as is the truth value algebra of interval-valued fuzzy sets [8,9]. Any subalgebra of **M** could serve as a basis of a fuzzy theory, a fuzzy set being a mapping of a universal set into this subalgebra. Thus subalgebras are of interest on two counts, as bases of fuzzy theories, and as mathematical entities per se.

The subalgebra of concern in this paper is the subset **E** of all mappings of the unit interval into the two-element set {0, 1}. Of course it is in natural one-to-one correspondence with the subsets of the unit interval, but the operations analogous to meet and join induced on this subalgebra do not correspond to the usual ones of intersection and union. We develop some of the basic properties of **E**, its automorphisms, and type-2 t-norms and t-conorms on it. In so doing, it will be conceptually and computationally convenient to view it as an algebra of subsets of [0, 1] with appropriate operations.

We begin in Section 2 with background material on the algebra **M** of fuzzy truth values, and some properties of the subalgebra **E**. In Section 3, we present the basic facts about the subalgebra **E**, including the identification of some special subalgebras and simplified methods of computing the basic operations of **E**. These lead to simplified proofs for **E** of results

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known for **M**, as well as to some new results for **E**. Section 4 is concerned with automorphisms of **E**, and we show that the automorphism group of **E** is naturally isomorphic to the automorphism group of the unit interval. In Section 5, we display homomorphisms between several of the algebras related to **E**. Finally, in Section 6 we consider properties of convolutions of t-norms operating on **E** and on the algebra of finite sets in **E**.

2. The algebra $\mathrm{M} = (\boldsymbol{M}, \sqcup, \sqcap, *, \bar{\mathbf{0}}, \bar{\mathbf{1}})$

A **type-2 fuzzy subset** of a set U is a mapping from U into the set $M = [0,1]^{[0,1]}$ of all mappings from the unit interval into itself. Operations on the set of all such fuzzy subsets of U come pointwise from operations on M. The basic operations of **M** were given by Zadeh [21], and we describe those now

Let $\wedge, \vee$, and $'$ be the usual operations on [0,1] given by

$$\begin{aligned} x \wedge y &= \min\{x, y\} \\ x \vee y &= \max\{x, y\} \\ x' &= 1 - x \end{aligned} \tag{1}$$

and let f and g be in $M = [0,1]^{[0,1]}$. The binary operations $\sqcup$ and $\sqcap$ on M are defined by the equations

$$\begin{aligned} (f \sqcup g)(x) &= \sup\{(f(y) \wedge g(z)) : y \vee z = x\} \\ (f \sqcap g)(x) &= \sup\{(f(y) \wedge g(z)) : y \wedge z = x\} \end{aligned} \tag{2}$$

and the unary $*$ operation by

$$f^*(x) = \sup\{f(y) : y' = x\} = f(x') \tag{3}$$

We denote by $\bar{1}$ and $\bar{0}$ the elements of M defined by

$$\begin{aligned} \bar{1}(x) &= \begin{cases} 0 & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1 \end{cases} \\ \bar{0}(x) &= \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases} \end{aligned} \tag{4}$$

At this point, we have the algebra $\mathbf{M} = (M, \sqcup, \sqcap, ^*, \bar{0}, \bar{1})$. This is the basic algebra for type-2 fuzzy set theory. Whatever equations this algebra satisfies will be automatically satisfied by the set of all fuzzy type-2 subsets of a set U with the corresponding pointwise operations.

The set M also has the pointwise operations $\vee, \wedge, '$ on it coming from operations on [0,1], and is a De Morgan algebra under these operations. In particular, under these operations, it is a lattice with order given by $f \leqslant g$ if $f = f \wedge g$, or equivalently, if $g = f \vee g$. These operations are useful in deriving properties of the algebra **M**. See [19]. For simplification of notation, we write $f \in \mathbf{M}$ to mean $f \in M$. In general, we adhere as much as possible to the notation in [19,20].

Theorem 1. *For $f \in \mathbf{M}$, let f^L and f^R be the elements of* **M** *defined by*

$$\begin{aligned} f^L(x) &= \vee_{y \leqslant x} f(y) \\ f^R(x) &= \vee_{y \geqslant x} f(y) \end{aligned} \tag{5}$$

Then for f and $g \in \mathbf{M}$

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L) \tag{6}$$

$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R) \tag{7}$$

Note that for any $f \in M, f^L$ is the smallest monotone increasing function above f and f^R is the smallest monotone decreasing function above f. Here, "smallest" is relative to the pointwise operations $\wedge$ and $\vee$ on M.

Using Eqs. (6),(7), it is fairly straightforward to verify the following basic properties of **M**. See [19] for details.

Corollary 2. *Let $f,\ g,\ h \in \mathbf{M}$. Then*

1. $f \sqcup f = f; \quad f \sqcap f = f$
2. $f \sqcup g = g \sqcup f; \quad f \sqcap g = g \sqcap f$
3. $\bar{1} \sqcap f = f; \quad \bar{0} \sqcup f = f$
4. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h; \quad f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
5. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
6. $f^{**} = f$
7. $(f \sqcup g)^* = f^* \sqcap g^*; \quad (f \sqcap g)^* = f^* \sqcup g^*$

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