International Journal of Approximate Reasoning 71 (2016) 64–88

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Finitely maxitive T-conditional possibility theory: Coherence and extension

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ARTICLE INFO

Article history:
Received 20 July 2015
Received in revised form 26 January 2016
Accepted 29 January 2016
Available online 2 February 2016

Keywords:
Finitely maxitive measure
T-conditional possibility
Coherence
Extension

ABSTRACT

Starting from the axiomatic definition of finitely maxitive T-conditional possibility (where T is a continuous triangular norm), the paper aims at a comprehensive and self-contained treatment of coherence and extension of a possibilistic assessment defined on an arbitrary set of conditional events. Coherence (or consistence with a T-conditional possibility) is characterized either in terms of existence of a linearly ordered class of finitely maxitive possibility measures (T-nested class) agreeing with the assessment, or in terms of solvability of a finite sequence of nonlinear systems for every finite subfamily of conditional events. Coherence reveals to be a necessary and sufficient condition for the extendibility of an assessment to any superset of conditional events and, in the case of T equal to the minimum or a strict t-norm, the set of coherent values for the possibility of a new conditional event can be computed solving two optimization problems over a finite sequence of nonlinear systems for every finite subfamily of conditional events.

1. Introduction

Since their introduction by Shilkret [61], maxitive measures proved to be a useful tool in many fields such as optimization theory [1,57], large deviations theory [2,54,58], and idempotent analysis [50,57].

In the realm of maxitive measures, we focus on *possibility measures*, also known as *idempotent probabilities* [50,58], which are normalized maxitive measures defined on a Boolean algebra $\mathcal{B}$. These measures have gathered a lot of attention in recent past, mainly due to their key role in fuzzy set theory and soft computing, which dates back to the pioneering paper by Zadeh [67] (see also [8,32,39,53,63], for a small account of the quite extended literature on the topic). As is well-known, every possibility measure Π induces a dual measure N, called *necessity measure*, defined as $N(E) = 1 - \Pi(E^c)$, for $E \in \mathcal{B}$.

Possibility theory is nowadays a well-established uncertainty framework that is sometimes considered as a "qualitative" alternative to probability theory. Actually there are several links between the two frameworks: for example, possibility measures can be characterized as upper envelopes of the class of all de Finetti's coherent extensions of a probability measure under suitable logical conditions (see [26,40]). Other probabilistic interpretations of possibility measures exist, such as the one in terms of random sets [51,52], the one in terms of large deviations [54,58], or the one in terms of likelihood functions [17,23–25,62].

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http://dx.doi.org/10.1016/j.ijar.2016.01.006
0888-613X/

If the algebra $\mathcal{B}$ is finite, then Π is completely characterized by its restriction on the atoms of $\mathcal{B}$, also referred to as *possibility distribution*. In order to maintain this characteristic, the classical definition of a possibility measure Π assumes that $\mathcal{B}$ is a complete atomic Boolean algebra and that Π is *completely maxitive*, i.e., it is a supremum preserving function [30,33,39,67].

In this paper, we consider a more general definition of possibility measure in which no assumption is made on the Boolean algebra $\mathcal{B}$ and Π is only asked to be *finitely maxitive* [61]. Notice that completely maxitive possibility measures constitute a distinguished subclass in this more general setting.

In the quantification of the uncertainty of an event E through an uncertainty measure φ, the main problem is to be able to assess the conditional measure $\varphi(E|H)$, which allows to take into account all the relevant "information" on E carried by some other event H, acting like a hypothesis. This is particularly important in inferential processes or for reasoning under (uncertain) hypotheses.

The concept of conditioning for possibility measures has been largely debated in the related literature, starting from the seminal works [8,46,53,67]. These papers have given rise to various definitions of conditional possibility (see, for instance, [31,39]), mainly introduced in analogy with probability theory or by means of suitable rules of conditioning. A common accepted definition of conditional possibility is still lacking so many (essentially semantic) justifications for the given definitions have been proposed.

In all definitions presented in the papers above, a conditional possibility is obtained as a derived concept from joint and marginal possibility distributions or measures. In particular, a large class of definitions considers $\Pi(E|H)$ as a solution of the equation $\Pi(E \wedge H) = T(x, \Pi(H))$, where T is a triangular norm [48]. Notice that the continuity of T only assures that the previous equation has always a solution, while uniqueness is not guaranteed. In order to get a unique solution further constraints must be imposed, such as the *minimum specificity principle* [39], which consists in always choosing the greatest solution.

Another alternative is to interpret a conditional possibility as a conditional upper probability, i.e., as the upper envelope of a particular class of conditional probabilities [18,36,41,64].

Nevertheless, a completely different approach is possible: a conditional uncertainty measure $\varphi(\cdot|\cdot)$ is axiomatically defined as a primitive concept, following the view outlined for probability theory by [34,35,37,49,60] and rediscovered starting from a suitable concept of truth value for conditional events [20,22].

The same axiomatic approach has been applied to conditional possibility [9,10,21] defining it as a function $\Pi(\cdot|\cdot)$ whose domain is a structured set of conditional events so that $\Pi(E|H)$ can be defined for any pair of events (E, H), with $H \neq \emptyset$, without restrictions caused by particular values of the joint and marginal unconditional possibilities.

Actually, there is an entire class of definitions parametrized by the choice of the t-norm T, which is the operation expressing the link among $\Pi(A|B)$, $\Pi(B|C)$ and $\Pi(A|C)$, when $A \subseteq B \subseteq C$. In fact, while for conditional probability the algebraic product is the only possible choice (as this operation must be distributive over the sum), for conditional possibility every t-norm can be considered since it distributes over the maximum.

The direct definition of T-conditional possibility enables us to obtain a general framework of reference in which most of the previous notions of conditional possibility can be seen as particular cases. Nevertheless, since the axioms work only under precise algebraic constraints on the domain, to handle partial assessments we need to consider the concept of *coherence* (i.e., consistence with the conditional measure of reference) and, in order to make it "operative", we have to characterize coherent assessments and their coherent extensions.

Inspired by the characterization of finitely additive conditional probabilities [12,19,49], we generalize to the infinite case the results obtained for finite domains in [9,10,27,29]. In detail, we provide a general characterization of T-conditional possibilities and coherent T-conditional possibility assessments (with T a continuous t-norm) in terms of existence of a linearly ordered class (T-nested class) of finitely maxitive possibility measures $\{\Pi_\alpha : \alpha \in \Gamma\}$ such that for every $E|H$ in the domain of $\Pi(\cdot|\cdot)$, there exists a unique $\alpha_H \in \Gamma$ for which $\Pi(E|H)$ is a solution of the equation $\Pi_\beta(E \wedge H) = T(x, \Pi_\beta(H))$ for $\beta \leq \alpha_H$, being the unique solution for α_H. Another equivalent characterization is then presented, relying on the solvability of a finite sequence of nonlinear systems for every finite subfamily of conditional events in the domain of $\Pi(\cdot|\cdot)$.

Moreover, we prove that a coherent assessment on an arbitrary family $\mathcal{G}$ of conditional events is coherently extendible (generally not in a unique way) to any superset of conditional events $\mathcal{G}'$ in the case T is the minimum or a strict t-norm. In particular, if $\mathcal{G}' = \mathcal{G} \cup \{E|H\}$ the problem of computing the set of coherent values for the possibility of $E|H$ is shown to be equivalent to two optimization problems over a finite sequence of nonlinear systems for every finite subfamily of $\mathcal{G}$.

Finally, we consider the problem of conditioning for necessity measures and show that a direct definition of conditional necessity is problematic, justifying in this way the main emphasis on conditional possibility adopted in this paper.

The paper is structured as follows. In Section 2 the axiomatic definition of T-conditional possibility is given. Section 3 copes with the representation of a T-conditional possibility defined on all the possible conditional events $E|H$ with $H \neq \emptyset$, while Section 4 considers the extendibility of some partially defined T-conditional possibilities. In Section 5 coherent T-conditional possibilities are introduced and a characterization of coherence and extendibility is provided. Sections 4 and 5 summarize for the sake of completeness some results that were proved in [13,56] and which have not yet appeared in any journal. Such results are completed and connected to new results of the present paper. Then, Section 6 faces the problem of conditioning for necessity measures, while Section 7 highlights some differences between coherent conditional probability theory and coherent T-conditional possibility theory. Finally, Section 8 draws some conclusions on the present work.