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Logics for Approximate Entailment in ordered universes of discourse

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A B S T R A C T

The Logic of Approximate Entailment (LAE) is a graded counterpart of classical propositional calculus, where conclusions that are only approximately correct can be drawn. This is achieved by equipping the underlying set of possible worlds with a similarity relation. When using this logic in applications, however, a disadvantage must be accepted; namely, in LAE it is not possible to combine conclusions in a conjunctive way. In order to overcome this drawback, we propose in this paper a modification of LAE where, at the semantic level, the underlying set of worlds is moreover endowed with an order structure. The chosen framework is designed in view of possible applications.

1. Introduction

In his seminal work on similarity-based reasoning [23], E. Ruspini proposes the interpretation of fuzzy sets in terms of (crisp) sets and fuzzy similarity relations. To this end, he builds up a framework for approximate inference that is based on the mutual similarity of the propositions involved. Following these lines, a number of approaches have dealt with similarity-based reasoning from a logical perspective [9–11,13]; see also [18, Section 5.2]. In particular, in the PhD thesis of R. Rodríguez [22], the so-called Logic of Approximate Entailment (LAE) is studied.

LAE is a propositional logic and propositions are interpreted, as in classical logic, by subsets of a fixed set, called the set of worlds. Propositions can be logically combined like in classical propositional logic and the Boolean connectives are interpreted by the corresponding set-theoretic operations as usual. However, it is in addition assumed that the set of worlds is endowed with a fuzzy similarity relation, which associates with each pair of two worlds their degree of resemblance. The basic semantic structures are hence *fuzzy similarity spaces*, which consist of a set of worlds and a fuzzy similarity relation, and the core syntactic objects of LAE are implications between propositions endowed with a degree. The intended meaning of a statement of the form $\alpha >_c \beta$ is that β is an approximate consequence of α to the degree c, where c is a real number between 0 and 1. If $c = 1$, the implication is defined to hold under the same condition as in classical propositional logic: at any world at which α holds, also β must hold. If $c < 1$, however, the statement is weaker, namely, we do not require in this case that if α holds at a world w, also β holds at w, we only require that there is a further world w' at which β holds and whose similarity with w is at least c. See Fig. 1 for an illustration.

Logics dealing with statements that are interpreted in metric spaces have been studied also from different points of view. Logics for spaces endowed with a metric or a more general distance function have been considered in a series of contri-

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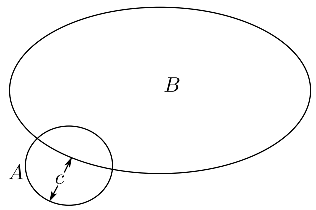

Fig. 1. The graded entailment in LAE. Let A and B be the sets of worlds at which α and β hold, respectively. Then $\alpha >_c \beta$ means that A is in the c-neighbourhood of B. Note that c varies between 0 and 1 and a smaller value of c corresponds to a greater distance.

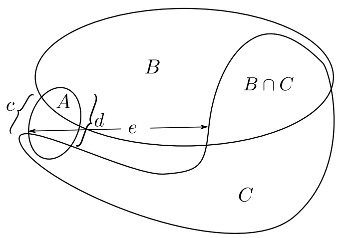

Fig. 2. The conjunction in LAE. If A is in the c-neighbourhood of B as well as in the d-neighbourhood of C, we cannot make any prediction about the value e such that A is in the e-neighbourhood of $B \cap C$.

butions by Kutz et al. and Sheremet et al. [16,17,24]. Furthermore, logics on comparative similarity have been studied by Alenda et al., see, e.g., [1–3]. It is also worth mentioning that there are some connections with graded or fuzzy consequence relations as studied by Pavelka [21,20], Chakraborty [6,7], and Gerla [12] among others in the context of many-valued logics, since indeed, graded implications $\alpha >_c \beta$ capture, at a syntactic (meta-)level, the idea of β being a consequence of α to the degree c. However, in the present context, α and β are classical propositions, not many-valued ones.

The starting point for the present paper is the aforementioned logic LAE. Although the concept underlying this logic is appealing, a disadvantage must be accepted. Deploying LAE in applications is difficult for a simple reason: in LAE we cannot combine conclusions in a conjunctive way. Assume that we have $\alpha >_c \beta$ and $\alpha >_d \gamma$, where $0 < c, d < 1$. Then we cannot in general derive in LAE a statement of the form $\alpha >_e \beta \wedge \gamma$ for some non-zero e. This feature of LAE is a straightforward consequence of the chosen semantic framework: if α implies that we are close to a situation in which β holds and moreover close to a situation in which γ holds, we cannot conclude that we are actually close to a situation in which both β and γ hold. In other words, for any sets of worlds A, B and C, if A is in the c-neighbourhood of B as well as in the d-neighbourhood of C, we cannot make any prediction about the value e such that A is in the e-neighbourhood of $B \cap C$. Refer to Fig. 2 for an illustration. In the extreme case, β and γ can even be contradictory. In such a case, there is no world at which both β and γ hold and $\beta \wedge \gamma$ will be interpreted by the empty set; but the e-neighbourhood of the empty set is empty for any e.

The lack of a rule that combines conclusions in a conjunctive way may be found restrictive in applications. Let us consider the following example; let the symbols α, β, γ denote the following properties of a car:

α "power (car) = 110 CV"
β "price (car) ≥ 20 000 €"
γ "consumption (car) ≥ 6 L/100 km"

Assume that our domain knowledge tells us that powerful cars are expensive to some extent and at the same time they have a high consumption. These facts could be reflected by a theory containing the graded implications

$$\alpha >_c \beta, \quad \alpha >_d \gamma, \tag{1}$$

where c and d are some appropriate non-zero degrees. It then seems natural to be able to derive $\alpha >_e \beta \wedge \gamma$ for some positive degree e.

This situation is certainly not appropriately reflected by Fig. 2. The crucial difference is the independence of the properties β, γ occurring in the conclusions. Price and consumption can indeed be assumed as not being interrelated. Consequently, a model can be based on a set of worlds consisting of all pairs of possible prices and possible consumption. Property α, the power of the car, is in turn assumed to have an influence on the other two. To reflect this influence, α is to be identified with those pairs of a price and a consumption that are not in contradiction with it. Assuming, for instance, that a power of 110 CV implies a price range between 15 000 € and 30 000 € as well as a petrol consumption between 5 L/100 km and 9 L/100 km, our model would be the one indicated in Fig. 3. Finally, a similarity between worlds can be computed as an aggregation of the similarities with regard to β and γ, like for instance their minimum. Under these assumption, we are able to derive from (1) the implication $\alpha >_{\min(a,b)} \beta \wedge \gamma$.