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The middle-parametric representation of fuzzy numbers and applications to fuzzy interpolation

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ABSTRACT

In this paper we introduce the middle-parametric representation of a fuzzy number presenting some of the advantages in the use of this representation. A special attention is focused on the subset of symmetric fuzzy numbers presenting the special properties of their arithmetic. The approach on symmetric fuzzy numbers is sustained by the applications of these kinds of fuzzy numbers in fuzzy linear programming and by the presence of the symmetric Gaussian type fuzzy numbers in the theory of errors. As potential applications of the middle-parametric representation, some fuzzy interpolation problems are considered.

1. Introduction

In what concerns fuzzy arithmetic, operations with fuzzy numbers are generated by using Zadeh's extension principle, but the existence of an opposite (with addition), or an inverse (with multiplication) for noncrisp fuzzy numbers is not warranted, what generates cautions in operating with fuzzy numbers. Moreover, the distributivity of the scalar product and the product of noncrisp fuzzy numbers is not valid everywhere (see for instance, [9,12,40], and [38]). In order to avoid this, some equivalence relations were proposed in [9,40,37,38], and [44], such that only on the obtained quotient structures the inversability holds.

In this context, for the distributivity of the scalar product, the following property holds:

$$(\alpha+\beta)\cdot u=\alpha\cdot u+\beta\cdot u,\quad \forall u\in\mathbb{R}_{\mathcal{F}} \text{ if and only if } \alpha\cdot\beta\geq 0.$$

This restriction can be avoided by using the additive equivalence introduced in [37] and obtaining an equivalence, not an identity (see [9,40], and [38]). Consequently, the additive structure of the set of fuzzy numbers is not a linear space, its appropriate structure being the cancellative quasilinear space described in [41]. The distributivity of the product of noncrisp fuzzy numbers holds generally only for fuzzy numbers with positive support.

If possible, in order to improve or to overcome some of the above presented difficulties arising in fuzzy arithmetic, we generalize the idea of using the "midpoint" of the core observed in [35] and propose a new representation of a fuzzy number $u\in\mathbb{R}_{\mathcal{F}}$, the *middle parametric representation*: $[\frac{u^-(t)+u^+(t)}{2},\frac{u^+(t)-u^-(t)}{2}]$, $t\in[0,1]$.

There are several representations for a fuzzy number. Now, we briefly present the existing representations of fuzzy numbers which have been done in the literature. Using the level sets of a fuzzy number, the lower-upper (LU) representation

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$[u_r^-, u_r^+]$, $r \in [0,1]$, is obtained in [29]. This is the widely used representation of a fuzzy number and was considered in [19] in order to introduce the notions of value and ambiguity. Using the LU-representation, the arithmetic operations with fuzzy numbers generated by the *extension principle* take the form presented in Equations (2), (3), (4). In this context, Dubois and Prade introduced in [21] and [22] the LR model for fuzzy numbers with corresponding analytical formulas of operations. The LR form of a fuzzy number is adequate from applications point of view. An extensive survey on fuzzy arithmetic and fuzzy intervals is realized in [24]. Starting from (LU) representation and using piecewise Hermite type monotonic interpolation (with polynomial or rational basis and with two double knots) in order to approximate the shape of a given fuzzy number, the so-called *parametric representation* of a fuzzy number is proposed in [30] and [47], providing an interesting procedure of approximating the arithmetic operations and presenting various applications to fuzzy calculus with shape-preserved results. This representation has the advantage of providing flexible and easy to control shapes of the fuzzy numbers in terms of a finite set of parameters, offering efficient algorithms of fuzzy calculus, but still remaining an approximation method. Consequently, the parametric representation from [30] and [47] is a suitable approximation of the well-known LU-representation. Another parametric approximation of a fuzzy number in LU representation is obtained in [11]. An interesting representation of fuzzy numbers was recently introduced in [27], the CE-representation (core-ecart) defined for $A \in \mathbb{R}_{\mathcal{F}}$, $A = [x_A^-(t), x_A^+(t)]$, $t \in [0,1]$, by considering $A = \left((a_1, \delta_A^-), (a_2, \delta_A^+)\right)$ where

$$a_1 = x_A^-(1),\ a_2 = x_A^+(1),\ \delta_A^-(t) = a_1 - x_A^-(t),\ \delta_A^+(t) = x_A^+(t) - a_2, \quad t \in [0,1].$$

In the CE-representation are involved the core $[a_1, a_2]$, the left and right "ecart" δ_A^-, and δ_A^+, respectively. This CE-representation is defined in order to introduce the Dorroh-type product $A \odot B = \left((a_1 b_1, \delta_{A\odot B}^-), (a_2 b_2, \delta_{A\odot B}^+)\right)$ where

$$\begin{cases} \delta_{A\odot B}^-(t) = a_1\delta_B^-(t) + b_1\delta_A^-(t) + \delta_A^-(t)\cdot\delta_B^-(t) \\ \delta_{A\odot B}^+(t) = a_2\delta_B^+(t) + b_2\delta_A^+(t) + \delta_A^+(t)\cdot\delta_B^+(t) \end{cases}, \quad t \in [0,1].$$

The above presented Dorroh-type product generalizes the cross-product defined in [10]. This product together with the usual addition lead to a semiring structure on the set of fuzzy numbers.

Some anticipatory ideas for the use of the middle point or middle function for fuzzy numbers or intervals appear in [14,16,26,32,35,44,45], and [46]. In [14], in order to construct a suitable distance between fuzzy numbers, it is proposed the following distance between two intervals $A = [a^{(1)}, a^{(2)}]$ and $B = [b^{(1)}, b^{(2)}]$

$$d^2(A,B) = k\left(a^{(1)} - b^{(1)}\right)^2 + h\left(a^{(G)} - b^{(G)}\right)^2 + k\left(a^{(2)} - b^{(2)}\right)^2$$

where $a^{(G)} = \frac{a^{(1)}+a^{(2)}}{2}$ and $2k + h = 1$, specifying the advantages of using this kind of distance. The following representation of a fuzzy number $u = (u_0, u_*, u^*)$ is proposed in [35], where

$$u_0 = \frac{u^-(1) + u^+(1)}{2},\ u_*(t) = u_0 - u^-(t),\ u^*(t) = u^+(t) - u_0, \quad t \in [0,1],$$

in order to improve the algebraic properties of operations with fuzzy numbers, where the corresponding arithmetic operations are defined by $u \circ v = (u_0 \circ v_0, u_* \vee v_*, u^* \vee v^*)$. Here $\circ$ is instead of $+, \cdot, -, /$. With this definition, the product of fuzzy numbers is defined everywhere on $\mathbb{R}_{\mathcal{F}}$ being fully distributive, but some information is lost by passing from operands to the result. The middle function $\frac{u_r^- + u_r^+}{2}$ is used in [32] in order to define the expected average of a fuzzy number

$$ea(u) = \int_0^1 \frac{\left(u^-(r) + u^+(r)\right)}{2} dr$$

and to be the representative of the equivalence classes with respect to the relation defined on $\mathbb{R}_{\mathcal{F}}$ by

$$u \equiv v \text{ iff } \forall r \in [0,1],\ \exists \varepsilon_r \in \mathbb{R} \text{ such that } [u_r^- - \varepsilon_r, u_r^+ + \varepsilon_r] = [v_r^-, v_r^+].$$

In [16], the midpoint $M(a)$ and the radius $R(a)$ of an interval $a = [a^-, a^+]$

$$M(a) = \frac{a^- + a^+}{2}, \quad R(a) = \frac{a^- - a^+}{2}$$

are used in fuzzy linear regression involving fuzzy intervals, with special attention for triangular and trapezoidal fuzzy intervals. The midpoint of each level set is considered in [45] and [46] so as to characterize the equivalence classes with respect to the additive congruence introduced by Mareš in [37]. Similarly, the quotient set of $\mathbb{R}_{\mathcal{F}}$ in relation to the same congruence is characterized in [26], through algebraic-topological isomorphism, by the group $BVC[0,1]$ of functions with bounded variation on $[0,1]$, improving the results from [15]. In this context, a representative of the equivalence class containing $u = [u_r^-, u_r^+]$, $r \in [0,1]$, is the arithmetic mean $m_a(u) = \frac{u^- + u^+}{2}$.