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Coherent updating of non-additive measures

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ABSTRACT

The conditions under which a 2-monotone lower prevision can be uniquely updated (in the sense of focusing) to a conditional lower prevision are determined. Then a number of particular cases are investigated: completely monotone lower previsions, for which equivalent conditions in terms of the focal elements of the associated belief function are established; random sets, for which some conditions in terms of the measurable selections can be given; and minitive lower previsions, which are shown to correspond to the particular case of vacuous lower previsions.

1. Introduction

The theory of imprecise probabilities contains a wide variety of mathematical models which are of interest in situations where it is unfeasible to determine, with certain guarantees, the probability model associated with an experiment. It includes for instance 2-monotone capacities [4], belief functions [43], possibility and necessity measures [24,56], random sets [17,41] or coherent lower previsions [48]. Under any of them, one important problem is that of updating the model under the light of new information. Unfortunately, this matter is far from settled, and quite a number of different rules has been proposed (see for instance [52] for an overview in the case of possibility measures). Among the most popular are Dempster's rule of conditioning [17,43], regular extension [9,29] and natural extension [48].

In order to be able to choose one rule above the others for a particular problem, it is essential to have a clear interpretation of the mathematical model we are using, and of what we mean by *updating* in our context. In this paper we deal with *epistemic* probabilities, where we model degrees of (partial) knowledge from a subject, and more specifically we focus on the behavioural approach championed by Peter Walley [48], that has its roots in the works on subjective probability by Bruno de Finetti [16]. This approach regards the lower and upper probabilities of an event as its supremum and infimum acceptable betting rates, and focuses on a consistency notion between these betting rates called *coherence*. Although this may seem restrictive at first, we argue that this is not the case, for a number of reasons:

- Imprecise probability models satisfying the notion of coherence (from now on *coherent lower previsions*) are always the envelopes of a convex set of probability measures. As a consequence, the behavioural approach is also compatible with a Bayesian sensitivity analysis interpretation.
- Almost all models of imprecise probabilities considered in the literature can be seen as particular instances of coherent lower previsions [51], and as a consequence our results shall be applicable also to them.

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With respect to the meaning of updating, although in the case of a precise probability model this is relatively straightforward, and it amounts to applying Bayes' rule, when we move to the imprecise case the situation becomes more complicated. There are basically two main scenarios:

- *Belief revision* [1,30], where we modify either the generic knowledge or the factual evidence about the problem under the light of new knowledge/evidence. The modification is usually done under the principle of minimal change. In the context of imprecise probabilities it gives rise to rules such as Dempster's rule of conditioning [17,27].
- *Focusing* [27, Section 6], where we condition our generic knowledge on factual evidence. This produces rules such as the regular extension [29].

Within Walley's behavioural approach to imprecise probabilities, the interpretation of the lower prevision of a gamble conditional on an event B corresponds to the *current* supremum acceptable betting rate we would establish for the gamble, assuming that later we come to know that the outcome of the experiment belongs to B. As such, it tells us which are the predictions associated with our current model, and therefore the process of updating corresponds to a problem of focusing.

Using this interpretation, Walley proposes in [48, Chapter 6] a notion of coherence that tells us if the conditional betting rates are compatible with the unconditional ones. However, this notion does not suffice to uniquely determine the conditional models from the unconditional ones. This was shown for instance in [37], where it was established that in general we may have an infinite number of conditional models compatible with the unconditional one, and that the smallest and greatest such models are respectively determined by the procedures called *natural* and *regular* extension, whose underlying differences we shall discuss later in the paper. Here, we investigate under which conditions the natural and the regular extensions coincide, and as a consequence there is only one conditional model that is coherent with the unconditional one. This would mean that in those cases it is not necessary to choose between the natural and the regular extensions (or any of other coherent rules that lie between them).

The rest of the paper is organised as follows: we shall recall the basics from the theory of coherent lower previsions in Section 2. Then we shall focus on a particular case of coherent lower previsions: those satisfying the property of 2-monotonicity [4,14]; these have the advantage that, unlike general coherent lower previsions, they are uniquely determined by their restrictions to events (a 2-monotone lower probability) by means of the Choquet integral.

After establishing a necessary and sufficient condition for the uniqueness of the coherent extensions in Section 3, we focus on two particular cases of 2-monotone lower previsions. First, in Section 4 we consider completely monotone lower previsions, which correspond to the Choquet integral with respect to a belief function [14]. We show that the necessary and sufficient condition mentioned above can be simplified by means of the focal elements of the belief function. Moreover, completely monotone lower previsions are associated with random sets, and from this we characterise the equality between the natural and the regular extensions in terms of the images of the random set; we also give an equivalent expression of the regular extension in terms of the measurable selections.

In Section 5 we focus on a second instance of 2-monotone lower previsions: the Choquet integral functionals with respect to Boolean necessity measures. Taking into account some recent results [12,13], these are related to the so-called *vacuous* lower previsions, which model a situation of complete ignorance about the outcome of an experiment. Interestingly, we show that both the natural and regular extensions also produce vacuous models, although they do not coincide in general; moreover, there exist also non-vacuous models coherent with our unconditional lower prevision.

One interesting fact stems from our results in this last section: that the problem of checking the coherence between the unconditional and the conditional models is not equivalent for lower probabilities and for lower previsions; and this even when the lower previsions, both in the unconditional and the conditional case, are uniquely determined by their associated lower probabilities. Indeed, in [52] it is proved that the smallest and greatest conditional possibility measures that are coherent with an unconditional possibility measure are the ones determined by Dempster's rule and by natural extension, respectively. This leads the authors to propose the harmonic mean between these two possibility measures as an updating rule. As we shall show, if we consider the upper previsions determined from these unconditional and conditional possibility measures by means of the Choquet integral, we obtain models that are not necessarily coherent.

We conclude the paper with some additional remarks in Section 6.

2. Preliminary concepts

Let us introduce the main concepts of the theory of coherent lower previsions we shall use in this paper. We refer to [48] for a more detailed exposition of the theory, and in particular of the behavioural interpretation of the concepts we shall introduce below. A survey of the theory can be found in [36].

2.1. Coherent lower previsions

Consider a possibility space Ω, that we shall assume in this paper to be *finite*. A *gamble* is a real-valued functional defined on Ω. We shall denote by $\mathcal{L}(\Omega)$ the set of all gambles on Ω. One instance of gambles are the indicators of events. Given a subset A of Ω, the indicator function of A is the gamble that takes the value 1 on the elements of A and 0 elsewhere. We shall denote this gamble by I_A, or by A when no confusion is possible.