International Journal of Approximate Reasoning 55 (2014) 130–141

Contents lists available at SciVerse ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

An axiomatic characterization of probabilistic rough sets

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ARTICLE INFO

Article history:
Available online 14 March 2013

Keywords:
Rough sets
Probabilistic rough sets
Approximation operators
Axioms

ABSTRACT

Probabilistic approaches were successfully applied to the theory of rough sets in recent decades. As a result, various types of probabilistic rough set models have been proposed in constructive approaches. This paper focuses on axiomatic approaches of probabilistic rough sets. Some new properties of probabilistic rough set approximation operators are examined in detail. By investigating the dependence among these properties, the axiom sets characterizing two types of the probabilistic rough set approximation operators are given. Each set of axioms guarantees the existence of an equivalence relation reproducing the corresponding probabilistic rough set approximation operators.

1. Introduction

Rough set theory proposed by Pawlak [15] is a powerful mathematical tool for processing uncertain and incomplete information. Using the basic concept of a pair of lower and upper approximation operators, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules.

By a thorough examination of studies on classical rough sets, Yao [31] suggested to classify them into the two classes of constructive and axiomatic approaches. The two classes are complementary to each other; each of them reflects a particular aspect of rough set theory and offers a distinctive view. It is very important to study rough set theory through both constructive and axiomatic approaches. In the constructive approaches [22,25,27,30,31], binary relations, partitions, and coverings of the universe of discourse, neighborhood systems and algebraic systems are primitive notions. Based on these notions the lower and upper approximation operators are constructed. On the other hand, the axiomatic approaches [20,26,27,31,39,40] take the abstractive set operators as primitive notions, these operators are characterized by many axioms, so that they can be constructed in constructive approaches. In terms of axiomatic approach, rough set theory may be interpreted as an extension of the classical set theory with two additional unary operators. For Pawlak approximation operators, Zakowski [38] and Comer [1,2] investigated axioms on approximation operators within the context of Pawlak information systems; Lin and Liu [11] suggested six axioms on rough approximation operators in the framework of topological spaces. For crisp binary relation-based rough approximation operators, the most important axiomatic studies have done by Yao et al. [30,31], where various crisp rough set algebras are characterized using different sets of axioms; Thiele [20] investigated axiomatic characterizations of approximation operators within modal logic.

As we know, Pawlak's rough set approaches can be used for classification problems [21]. Given a subset of the universe, all objects of the universe can be divided into three classes: the positive, negative, and boundary regions. This kind of classification relies on standard inclusion relation, the degree of set overlap was not considered. Namely, the classification must be totally correct or certain. Thus some useful knowledge hidden in the boundary region may not be fully and effectively used. In order to overcome the shortage, many researchers introduce probabilistic approaches to Pawlak rough set theory,

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0888-613X/$ - see front matter
http://dx.doi.org/10.1016/j.ijar.2013.02.012

as generalizations of Pawlak rough sets, various probabilistic rough set models have been proposed [12,33,34], such as the variable precision rough set model (VPRSM) [8,41,42], the decision-theoretic rough set model (DTRSM) [5,7,9,10,28,32,36], the Bayesian rough set model (BRSM) [3,13,17–19], and the game-theoretic rough set model (GTRSM) [4,6,29]. The four types of probabilistic rough set models, VPRSM, DTRSM, BRSM, and GTRSM can be formulated based on conditional probabilities, and the corresponding approximation operators have a similar formulation as in [33].

VPRSM [42] is one of the most important probabilistic rough set models, in which, the standard inclusion relation in Pawlak rough set model is extended to a generalized inclusion relation. The extended inclusion relation allows for some degree of misclassification, so more general association decision rules including deterministic and probabilistic ones can be obtained. In addition, Ziarko et al. [8] put forward an asymmetric variable precision rough set model (AVPRSM), the model becomes more flexible and applicative. Since VPRSMs with different parameters may induce distinct decision rules, how to choose parameters of approximations in the applications is a crucial issue. As for this, when the practical decision problems involve cost or risk environments, the DTRSM proposed by Yao et al. [36] presents an approach, by which, not only can the parameters in the model be computed directly, but also the semantic meaning of the model can be understood clearly [35,37]. Therefore, DTRSM establishes a solid theoretical foundation for probabilistic rough set models. It should be noted that the Pawlak's rough set model and VPRSM can be derived from DTRSM under relevant loss functions [32], and VPRSM can be considered as a result when using the decision theoretic approach for rough analysis [6]. On the other hand, GTRSM proposed by Herbert and Yao [4,6] presents another approach for optimizing parameters of DTRSM. GTRSM uses game theory to optimize the parameters, as a result, the minimum conditional probabilities that an object must be included in the positive, negative, or boundary regions can be determined, or the size of the boundary region can be decreased. For multi-agent environment, Yang and Yao [28] proposed an extended DTRSM called multi-agent DTRS, by which the decision making involving various needs of users can be simplified. Based on Bayesian reasoning, Slezak and Ziarko [19] presented another probabilistic rough set model, i.e. BRSM. The parameters in BRSM are determined referencing to the prior probability of of occurrence of the target events. As the prior probability can be derived only from the information provided by data set itself, comparing to the other versions of probabilistic rough set models, BRSM is more objective and useful.

All these existing studies on probabilistic rough sets are based on constructive approaches. They focus on building probabilistic approximations by using a pair of thresholds on conditional probabilities. There is a lack of a study based on axiomatic approaches. The objective of this paper is to fill in the gap by proposing an axiomatic approach to probabilistic rough sets. The results not only offer a different view but also lead to a new understanding of probabilistic rough sets. In this paper, we concentrate on providing sets of axioms characterizing two types of probabilistic rough set approximation operators. The detailed work is organized as follows. In Section 2, some notions and properties of Pawlak rough sets and probabilistic rough sets are reviewed. In Section 3, some new properties of probabilistic rough sets are examined in detail. In Section 4, by investigating the dependence among these properties, two axiom sets characterizing probabilistic rough sets are obtained. Finally, Section 5 concludes the paper.

2. Preliminaries

Let U be a finite and non-empty set called the universe of discourse, and E an equivalence relation on U, that is, E is reflexive, symmetric, and transitive. Then the pair (U, E) is called an approximation space. The basic knowledge in an approximation space (U, E) is the equivalence classes of E, For any $x \in U$, the equivalence class containing x is defined by

$$[x] = \{y \in U | (x, y) \in E\}.$$

The set of all equivalence classes is a partition of U, denoted by U/E, which is a family of pairwise disjoint subsets whose union is U. In the rough set theory, equivalence classes in U/E are understood as elementary definable, or measurable sets in the approximation space (U, E), and unions of elementary definable sets as definable sets. The family of all definable sets is a σ-algebra over U, denoted as $\sigma(U/E)$, that is, it contains the empty set $\emptyset$, U, and is closed with respect to set complement, intersection, and union.

Let (U, E) be an approximation space. For a subset $A \subseteq U$, the lower approximation and upper approximation of A are defined as follows [15]:

$$\underline{E}(A) = \{x \in U | [x] \subseteq A\}, \tag{1}$$

$$\overline{E}(A) = \{x \in U | [x] \cap A \neq \emptyset\}. \tag{2}$$

The following equivalent definitions are often used in the study of rough set theory [30]:

$$\underline{E}(A) = \bigcup \{[x] \in U/E | [x] \subseteq A\}, \tag{3}$$

$$\overline{E}(A) = \bigcup \{[x] \in U/E | [x] \cap A \neq \emptyset\}. \tag{4}$$

Eqs. (3) and (4) show that the lower approximation is the union of equivalence classes included in A, and the upper approximation is the union of equivalence classes having non-empty intersection with A.