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On fuzzy-qualitative descriptions and entropy

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ABSTRACT

This paper models the assessments of a group of experts when evaluating different magnitudes, features or objects by using linguistic descriptions. A new general representation of linguistic descriptions is provided by unifying ordinal and fuzzy perspectives. Fuzzy-qualitative labels are proposed as a generalization of the concept of qualitative labels over a well-ordered set. A lattice structure is established in the set of fuzzy-qualitative labels to enable the introduction of fuzzy-qualitative descriptions as L-fuzzy sets. A theorem is given that characterizes finite fuzzy partitions using fuzzy-qualitative labels, the cores and supports of which are qualitative labels. This theorem leads to a mathematical justification for commonly-used fuzzy partitions of real intervals via trapezoidal fuzzy sets. The information of a fuzzy-qualitative label is defined using a measure of specificity, in order to introduce the entropy of fuzzy-qualitative descriptions.

1. Introduction

One of the current challenges in knowledge representation and knowledge-based systems for decision making is the use of qualitative descriptions of variable values. This becomes necessary when numerical measurements of variables are unavailable, or when they are not convenient. In these cases, linguistic descriptions are used to represent uncertainty, as well as different levels of precision [5,6,10,16,21]. These types of systems have been used widely in engineering, as well as in biological, medical, economic, and social science applications, and recent examples can be found in [12,19].

Two main areas of linguistic information representation can be found in the literature [18]. Some approaches use fuzzy representations of linguistic descriptions [16,17]. On the other hand, some approaches use ordinal models and do not make an effective use of membership functions, being based either on 2-tuple modeling [5,10] or on order-of-magnitude qualitative models [20,25,26]. Methodologies involving different levels of precision during linguistic modeling can be found in both main areas. In the case of fuzzy approaches, they usually rely on a hierarchy defined from a fuzzy partition of a real interval by means of triangular or trapezoidal fuzzy numbers [6,16,17]. Approaches based on 2-tuple modeling consider a linguistic hierarchy to deal with different levels of precision [5,10]. Approaches based on absolute order-of-magnitude qualitative models use different levels of precision or abstraction in linguistic modeling by means of qualitative labels that in some cases correspond to sub-intervals coming from a partition of a real interval [25,26]. Furthermore, the concept of entropy was formalized to measure the amount of information both in fuzzy and in ordinal research areas [1,20].

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On the other hand, there are situations where uncertainty applies, not only to the lack of numerical knowledge of the values of a variable, but also to the selection of the linguistic terms describing such values [13,24]. To manage these situations, new fuzzy models were developed. For example, type-2 fuzzy sets were introduced as fuzzy sets whose membership grades are themselves fuzzy sets [9,11,13,14], and other fuzzy models, such as discrete interval type-2 fuzzy sets and hesitant fuzzy sets, consider a set of possible values when defining the membership of an element [9,21,24,28]. In addition, entropy has been studied in several fuzzy set theory extensions in recent literature, for instance, in hesitant, intuitionistic, type-2 and interval valued fuzzy sets [7,23,27,30].

This paper presents a mathematical contribution to the area of decision making. It models the assessments of a group of experts when evaluating different magnitudes, features or objects by using linguistic descriptions. In addition, it proposes a measure of the amount of information delivered by the different experts in these group decision-making processes. A new general representation of linguistic descriptions is provided by unifying ordinal and fuzzy perspectives.

Fuzzy-qualitative labels are introduced as fuzzy sets over a set S, whose elements can be associated with linguistic terms, by extending the model proposed in [20]. The set of fuzzy-qualitative labels is structured as a lattice, which enables us to introduce fuzzy-qualitative descriptions as L-fuzzy sets [3,15]. Moreover, fuzzy-qualitative descriptions are an extension of type-2 fuzzy sets, replacing in the secondary domain the unit interval with the set S [9,13]. Fuzzy-qualitative descriptions can also be considered as fuzzy random variables interpreted under the ontic model [2,4]. We formally introduce the concept of entropy of a fuzzy-qualitative description Q of a set Λ as the entropy associated with the probability measure induced by Q based on a measure on the power set $\mathcal{P}(\Lambda)$. The concept of entropy is then formalized by means of a Lebesgue integral and a measure of specificity [30]. In the discrete case, where Q has a finite range, this integral becomes a weighted average of the information of the labels, which corresponds to the Shannon self-information entropy of a discrete random variable. This concept allows the measurement of the amount of information given by a fuzzy-qualitative description and a comparison of expert assessments in group decision making [22]. In addition, a theorem is given that characterizes the finite fuzzy partitions of a well-ordered set using fuzzy-qualitative labels, leading to a full mathematical justification for the commonly used fuzzy partitions of real intervals via trapezoidal fuzzy numbers [16,17].

The remainder of this paper is organized as follows. In Section 2 the concept and structure of the set $\mathcal{L}$ of fuzzy-qualitative labels are introduced. Section 3 provides a characterization of the fuzzy partitions of a well-ordered set under certain conditions. The fuzzy-qualitative descriptions of a set Λ and the concepts of information and entropy are defined in Section 4. Finally, our conclusions and future research directions are presented in Section 5.

2. Fuzzy-qualitative labels over a well-ordered set $\mathcal{S}$

In this section, the concept of fuzzy-qualitative labels over a well-ordered set $\mathcal{S}$ is presented. This enables us to introduce fuzzy sets into order-of-magnitude qualitative reasoning [25]. Firstly, we provide a brief summary of some necessary concepts related to crisp qualitative labels introduced in [20].

2.1. Qualitative labels over a well-ordered set [20]

Given a well-ordered set $(\mathcal{S}, \leq)$, its singletons $\{a\}$, $a \in \mathcal{S}$, are considered to be *basic qualitative labels* (or *basic labels*) over $\mathcal{S}$. The *qualitative labels* (or *labels*) over $\mathcal{S}$ are the intervals $[a, b) = \{x \in \mathcal{S} \mid a \leq x < b\}$, for all $a, b \in \mathcal{S}$ with $a < b$, together with the intervals $[a, \rightarrow) = \{x \in \mathcal{S} \mid a \leq x\}$, for all $a \in \mathcal{S}$. In particular, the entire set $\mathcal{S} = [p, \rightarrow)$, where p is the least element of $\mathcal{S}$, is a label, and the basic labels are labels: $\{a\} = [a, s(a))$, where $s(a)$ is the successor of a, except in the case in which a is the last element of $\mathcal{S}$, if it exists, and then $\{a\} = [a, \rightarrow)$. In general, the label $\mathcal{S}$ is denoted by the symbol?, which is referred to as the *unknown label.*

The set $\mathbb{L}^*$ of all the qualitative labels over $\mathcal{S}$ is named the *order-of-magnitude space over* $\mathcal{S}$:

$$\mathbb{L}^* = \{[a, b) \mid a, b \in \mathcal{S},\ a < b\} \cup \{[a, \rightarrow) \mid a \in \mathcal{S}\}.$$

Note that $\mathbb{L}^* \subseteq \mathcal{P}(\mathcal{S})$, where $\mathcal{P}(\mathcal{S})$ is the power set of $\mathcal{S}$.

The set $\mathbb{L} = \mathbb{L}^* \cup \{\emptyset\}$ is named *the extended set* $\mathbb{L} \subseteq \mathcal{P}(\mathcal{S})$ *of qualitative labels over* $\mathcal{S}$, and $(\mathbb{L}, \sqcup, \cap)$ is a lattice, with the mix operation $\sqcup$ and the set intersection $\cap$.

2.2. Fuzzy-qualitative labels

This subsection presents a formal generalization of the order-of-magnitude space $\mathbb{L}^*$ over a well-ordered set $\mathcal{S}$ to a fuzzy framework. Fuzzy qualitative labels are defined as fuzzy sets with core qualitative labels, as follows.

Definition 1. A *fuzzy-qualitative label over* $\mathcal{S}$ is a fuzzy set $A \in \mathcal{F}(\mathcal{S}) = [0, 1]^{\mathcal{S}}$ such that $Core(A) \in \mathbb{L}^*$.

In this manner, $Core(A) = [a, b)$ or $Core(A) = [a, \rightarrow)$, for some $a, b \in \mathcal{S}$, $a < b$. In other words, the fuzzy-qualitative labels are the fuzzy sets on $\mathcal{S}$ for which the set of elements that belong to them with membership value equal to 1 is a qualitative label of $\mathbb{L}^*$.