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Multivariate dependence concepts through copulas

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ABSTRACT

In this paper, multivariate dependence concepts such as affiliation, association and positive lower orthant dependent are studied in terms of copulas. Relationships among these dependent concepts are obtained. An affiliation is a notion of dependence among the elements of a random vector. It has been shown that the affiliation property is preserved using linear interpolation of subcopula. Also our results are applied to the multivariate skew-normal copula. As an application, the dependence concepts used in auction with affiliated signals are discussed. Several examples are given for illustration of the main results.

1. Introduction

With the rapid development of mathematical finance and risk management in the last two decades, more and more attention has been paid to creating some practical statistical models beyond normal settings to improve competitive performance in finance and insurance fields. The copula is one of the most important models used in mathematical finance. Specifically, copulas, introduced in [17], are used to model multivariate data as they account for the dependence structure and provide a flexible representation of the multivariate distribution. Copulas are distributions with $[0,1]$-uniform marginals, which contain the most multivariate dependence structure properties and do not depend on the marginals. For references, see [5,11,13].

Auctions are popular mechanisms for selling products and constitute an important activity in economics. Nowadays, almost any goods, such as pieces of art, timber rights, used cars, treasury bills, can be sold through different types of auctions all over the world. The more current research trend in analysis of auction theory is called the common value (CV) auctions, where bidders make independent estimates of a common value (e.g., the unknown amount of mineral). In the framework of common value auctions, the dependence of bidder's signals is modeled by affiliation, which means that a high value of one bidder's estimate makes high values of the other's estimates more likely [9]. The affiliation is a concept from the multivariate statistics analysis. It is Milgrom and Weber [9] who first introduced the affiliation into the auction literature. The affiliation concept is important to auction from both theoretical and empirical viewpoints. In similar situations in econometrics, when dependence of random variables is a concern, the theory of affiliated copulas, which will be defined

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in next section, offers an appropriate approach for constructing auction models. Recently, Rinotta and Scarsini [15] studied the total positivity order for multivariate normal distributions. The importance of the affiliation properties in application of auction theory can be found in [4,10,14,16,18].

As an extension of normal settings, multivariate skew normal distributions are widely used in almost all fields for almost three decades. For references on skew normal distributions, see [1,2,20], and many other papers listed in the website of Azzalini [3]. The concept of affiliation on the class of multivariate skew normal family has not been investigated in the literature.

This paper is an extension of Wei et al. [21] from bivariate case to multivariate case. Dependence and association concepts as well as their relationships are obtained in Section 2. The affiliation and positive lower orthant dependent properties of a subcopula if preserved under linear interpolation are studied in Section 3. For illustrations, in Section 4, our results are applied to the family of skew-normal copulas. In Section 5, applications of the affiliation in auction are discussed.

2. Basic concepts

Following the notions of [15,19], for column vectors $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, $\boldsymbol{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, let $\boldsymbol{x} \vee \boldsymbol{y} = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})^T$ and $\boldsymbol{x} \wedge \boldsymbol{y} = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})^T$, where $\boldsymbol{x}^T$ is the transpose of $\boldsymbol{x}$.

Definition 2.1. The random vector $\boldsymbol{X} \in \mathbb{R}^n$ is said to be **affiliated** (or **positively likelihood ratio dependent** (PLRD)) if

$$h(\boldsymbol{x})h(\boldsymbol{y}) \le h(\boldsymbol{x} \vee \boldsymbol{y})h(\boldsymbol{x} \wedge \boldsymbol{y}) \tag{2.1}$$

holds for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$, where $h(\boldsymbol{x})$ is the probability density function (PDF) of $\boldsymbol{X}$.

In order to investigate the affiliation concept in terms of copula, we need the following definition of subcopula [11]:

Definition 2.2. An n-dimensional **subcopula** (or n-**subcopula**) is a function $C' : \prod_{i=1}^{n} S_i \mapsto [0, 1]$, where S_i's are subsets of $[0, 1]$ containing 0 and 1, with the following properties:

(a) C' grounded, i.e., if at least one $u_i = 0$, $C'(u_1, \cdots, u_n) = 0$;

(b) For $u_i \in [0, 1]$, $i = 1, \ldots, n$,

$$C'_i(u_i) \equiv C'(1, \cdots, 1, u_i, 1, \cdots, 1) = u_i;$$

(c) C' is n-increasing in the sense that, for any $J = \prod_{i=1}^{n} [u_i, v_i] \subseteq [0, 1]^n$, with u_i and $v_i \in S_i$, for $i = 1, \ldots, n$.

$$volC'(J) = \sum_{\boldsymbol{a}} sgn(\boldsymbol{a}) C'(\boldsymbol{a}) \ge 0,$$

where the summation is over all vertices $\boldsymbol{a}$ of J, and for $\boldsymbol{a} = (a_1, \ldots, a_n)^T$, with $a_i = u_i$ or v_i,

$$sgn(\boldsymbol{a}) = \begin{cases} 1, & \text{if} \quad a_i = v_i \quad \text{for an even number of } i\text{'s}, \\ -1, & \text{if} \quad a_i = v_i \quad \text{for an odd number of } i\text{'s}. \end{cases}$$

An n-**copula** $C : [0, 1]^n \to [0, 1]$ is a subcopula C' with $S_i = [0, 1]$, for all $i = 1, \ldots, n$.

Sklar's theorem [17] states that if H is the joint distribution (CDF) function of $\boldsymbol{X}$, then H evaluated at $\boldsymbol{x}$ can be expressed as $H(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n))$, where C is the copula corresponding to $H(\boldsymbol{x})$ and F_i is the CDF of X_i, $i = 1, \ldots, n$. A copula C characterizes dependence structures and dependence measures of $H(\boldsymbol{x})$, which is independent of marginal distributions. It can be viewed as a joint distribution of the random variables U_i on $[0, 1]$, $i = 1, \ldots, n$. Motivated by this, we give the corresponding affiliation definition for a copula as follows.

Definition 2.3. A copula $C(u_1, \cdots, u_n)$ is said to be **affiliated** if

$$c(\boldsymbol{u})c(\boldsymbol{v}) \le c(\boldsymbol{u} \vee \boldsymbol{v})c(\boldsymbol{u} \wedge \boldsymbol{v}) \tag{2.2}$$

holds for all $\boldsymbol{u} = (u_1, \ldots, u_n)^T$ and $\boldsymbol{v} = (v_1, \ldots, v_n)^T$ in $[0, 1]^n$, where $c(u_1, \cdots, u_n)$ is the PDF of $\boldsymbol{U} = (U_1, \ldots, U_n)^T$ corresponding to copula $C(u_1, \cdots, u_n)$ with $c(u_1, \cdots, u_n) = \frac{\partial^n C(u_1, \cdots, u_n)}{\partial u_1 \ldots \partial u_n}$.

Remark 2.1. It is true that the random vector $\boldsymbol{X}$ is affiliated if and only if its corresponding copula is affiliated. Indeed, suppose $\boldsymbol{X}$ is affiliated. Let $h(\boldsymbol{x})$ and $c(\boldsymbol{u})$ be the corresponding PDF of $\boldsymbol{X}$ and copula PDF of $\boldsymbol{U}$, respectively. Then

$$h(x_1, \cdots, x_n) = c(F_1(x_1), \cdots, F_n(x_n)) f_1(x_1) \cdots f_n(x_n),$$