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On finitely-valued Fuzzy Description Logics ☆

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ABSTRACT

This paper deals with finitely-valued fuzzy description languages from a logical point of view. From recent results in Mathematical Fuzzy Logic and following [44], we develop a Fuzzy Description Logic based on the fuzzy logic of a finite *BL*-chain. The constructors of the languages presented in this paper correspond to the connectives of that logic (containing an involutive negation, Monteiro–Baaz delta and hedges). The paper addresses the hierarchy of fuzzy attributive languages; knowledge bases and their reductions; reasoning tasks; and complexity. Our results regarding decidability together with a summary of the known results related to computational complexity are of particular interest. In Appendix B we also provide axiomatizations for expansions of the logic of a finite *BL*-chain considered in the paper.

1. Introduction

The last ten years has seen much and increasing interest in the attempt to generalize the formalism of Description Logics (DLs) to a multi-valued framework. In the literature there are several interesting and in-depth papers on Fuzzy Description Logics (FDLs) that deal with the expressiveness of the languages and reasoning algorithms (see [51] for a survey) rather than with logical foundations. In this paper, though, we take a metamathematical point of view and define FDLs on the basis of first-order many-valued fuzzy logics in an analogous way to how DLs relate to first-order classical logic.[1] As a result of the development of Mathematical Fuzzy Logic (MFL), we have at our disposal a large family of first-order logical systems, the so-called predicate t-norm-based fuzzy logics. These systems, presented as well-defined Hilbert-style calculi, allow us to interpret FDLs in them and, therefore, to take advantage of the results and metamathematical tools developed in Fuzzy Logic over the last fifteen years. A good reference for the field of MFL is [35] (see also http://www.mathfuzzlog.org for an exhaustive list of works and researchers in this area).

This approach to dealing with FDLs was first proposed by Hájek (see [44–46]) and has recently also been developed in [39] where some lines of research in the same direction are proposed. The main idea is to work in a similar way as in classical DLs whose formulas are interpreted into first-order classical logic. Consequently, we define FDLs (with the constructors needed to have the expressive capabilities we want to confer to the description language) and we interpret the formulas in the corresponding first-order fuzzy logic. In order to proceed with this agenda, we need to know the fragments of the logics corresponding to our FDLs, their properties, and their reasoning capabilities including complexity

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[1] Note that, following this approach we only deal with constructors coming from logic and we do not consider some of the subjects that are studied in papers about FDLs dealing with fuzzy set theoretical notions in a broader sense—such as fuzzy quantifiers [56] or fuzzy data types [63,9].

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and algorithms. The problem is that first-order fuzzy logics and their fragments are more complex and have been studied less than classical ones. For example, we know that many first-order fuzzy logics (such as Łukasiewicz or product logics) are not complete with respect to the semantics defined on $[0,1]_*$, i.e., on the structure defined on $[0,1]$ by a t-norm $*$ and its residuum (see [34]), while the semantics needed for defining FDLs is that on $[0,1]_*$. In particular, $[0,1]_*$-tautologies are very complex. For instance, they are not recursively enumerable in the case of Łukasiewicz logic and not arithmetical in the case of product logic (see [47] and references therein). Even though for Łukasiewicz (see [44]) and product (see [28]) logics, it is proved that the satisfiability problem for the fragment of the first-order fuzzy logic associated with the $\mathcal{ALC}$-like Fuzzy Description Logic (with respect to an empty knowledge base) is decidable, many other problems remain open.

In [39] a family of description languages is defined: one language for each continuous t-norm (or divisible finite t-norm) and each countable subalgebra of the corresponding standard algebra on the real interval $[0,1]$ (or on a finite chain $X_n = \{0 = r_1 < r_2 < \cdots < r_{n-1} < r_n = 1\}$). These languages, denoted in that paper by $\mathcal{ALC}^*(S)$, include an involutive negation and truth constants for representing truth degrees: one for each element of the carrier S of the subalgebra. In FDLs the so-called *graded formulas* are often used, which demand, from the logical side, an explicit representation of the truth values in the underlying logic. Taking advantage of the expressive power provided by these truth constants, in [39] a graded notion of satisfiability (subsumption) is defined by means of the satisfiability (validity) of certain formulas with truth constants of the associated first-order fuzzy logic, denoted by $\tilde{L}^*(S)\forall$ in that paper.

Finitely-valued FDLs show much potential for different applications and a good test for a study of reasoning algorithms. This is due to the fact that, in the presence of knowledge bases, many of the graded reasoning tasks in the setting of finitely-valued chains can be proved to be decidable, while in the case of infinite chains this is not always the case (see [2] for Product and [31] for Łukasiewicz). Moreover, the different graded satisfiability and subsumption notions that can be defined in the general FDL framework can be handled much more easily in the finite case than in the infinite case. Bearing these ideas in mind, this paper is devoted to developing FDLs over finite *BL*-chains from the logical point of view. The main features of the paper are:

(a) the introduction of hedges in the logical setting as unary connectives,
(b) the study of the hierarchy of FDL languages (with logic-based constructors),
(c) reduction among reasoning tasks in this finite framework, and
(d) the extension of Hájek's algorithm (which does not consider the presence of inclusion axioms) to the reasoning tasks for the finitely-valued FDLs studied in this paper.

Moreover, the paper brings together several results from research into this topic that are known but are disperse in the literature.

Let us just mention that fuzzy modifiers have already been introduced in FDL in several papers (see for example [66,64,48,59,62,26] and the review [52]) but, in the current paper we present them as logic-based constructors in accordance with recent results regarding the MFL framework (see [43,65,38]). To the best of our knowledge, this is the first time that hedges (either truth-stressers or depressers) are introduced as unary operations on the canonical chain following the general study of hedges in fuzzy logic, mainly in [38].

The paper begins with an example that shows the utility of using all the expressive power of t-norm-based fuzzy logics to define FDL languages. In the study we restrict ourselves to the case in which $*$ is a divisible finite t-norm, and we take as the algebra of truth values the finite standard algebra $\mathbf{C}_*$ defined on the set X_n by $*$ and its residuum $\rightarrow_*$, and enriched with some additional operators, such as an involutive negation (if the one defined as $\neg x = x \rightarrow_* 0$ is not involutive), the delta operator, and the unary operators associated with hedges. Thus, after a preliminary section devoted to some basic results concerning divisible finite t-norms, in Section 4 we introduce the logics $\Lambda_*\forall$ and $\Lambda_*[k,m]\forall$, which are the logical framework for the n-graded FDLs considered in this paper. In Section 5 we use the notion of an *instance* of a description in order to define the new family of n-graded FDLs as fragments of the aforementioned logics. Therefore, as in the classical case, our n-graded FDLs are related to fragments of the corresponding predicate calculi. In the same section, we also discuss the role of the constructor for *implication* in FDLs, and we define a hierarchy of description languages from the less expressive $\mathcal{AL}$-like to the more expressive $\mathcal{ALC}$-like adapted to the behavior of the connectives in the fuzzy logics underlying these description languages. In Section 6 we discuss reasoning tasks within the framework of finitely-valued logics; the different notions of graded satisfiability and subsumption and their relationships.

In recent years some results have been published that concern decidability and computational complexity for FDLs over a finite algebra of truth values. In Section 7 we summarize the state of the art with respect to the results that appear in the recent literature. A very interesting question is that of the decidability and computational complexity of FDLs over a finite *BL*-chain. To the best of our knowledge, the results regarding computational complexity seem to indicate that in general it does not change when shifting from bi-valued to finite-valued chains. This is a very interesting feature for applications in the sense that we can use the more expressive n-graded languages instead of the two-valued (crisp) languages while remaining in the same class of complexity.

The paper ends with the conclusion section and two appendices. Appendix A provides an example that explains how the reduction provided in [44], as well as that provided in [28], is not polynomial. Appendix B contains explicit axiomatizations for the fuzzy logics underlying the fuzzy description languages studied here.