International Journal of Approximate Reasoning 55 (2014) 639–653

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

A simple logic for reasoning about incomplete knowledge

Mohua Banerjee [a], Didier Dubois [b,c,*]

[a] Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India
[b] IRIT, CNRS Université de Toulouse, France
[c] School of Electronics, Electrical Engineering and Computer Science, Queens University, Belfast, UK

ARTICLE INFO

Article history:
Received 18 April 2013
Received in revised form 11 November 2013
Accepted 12 November 2013
Available online 13 November 2013

Keywords:
Incomplete information
Epistemic logic
Evidence theory
Possibility theory

ABSTRACT

The semantics of modal logics for reasoning about belief or knowledge is often described in terms of accessibility relations, which is too expressive to account for mere epistemic states of an agent. This paper proposes a simple logic whose atoms express epistemic attitudes about formulae expressed in another basic propositional language, and that allows for conjunctions, disjunctions and negations of belief or knowledge statements. It allows an agent to reason about what is known about the beliefs held by another agent. This simple epistemic logic borrows its syntax and axioms from the modal logic *KD*. It uses only a fragment of the *S*5 language, which makes it a two-tiered propositional logic rather than as an extension thereof. Its semantics is given in terms of epistemic states understood as subsets of mutually exclusive propositional interpretations. Our approach offers a logical grounding to uncertainty theories like possibility theory and belief functions. In fact, we define the most basic logic for possibility theory as shown by a completeness proof that does not rely on accessibility relations.

1. Motivation

Reasoning about knowledge and beliefs requires more than the language of classical propositional logic. In the syntax of classical propositional logic, it is only possible to express that certain propositions are known or believed. A set of logical formulae is then often called a knowledge base, or a belief base [42], and when it is deductively closed, a belief set [26]. The latter is used in belief revision for representing the dynamics of knowledge upon receiving new information. However, stating that some propositions are acknowledged as being unknown to an agent requires the use of a more expressive language, since the language of classical propositional logic cannot really express the difference between statements like "not knowing α" and "knowing not α" (in fact it can only express the latter as $\neg\alpha$). This distinction can only be made in the metalanguage (interpreting believing or knowing as proving [18]). In modal logic, the first statement writes $\neg\Box\alpha$, and the second one is $\Box\neg\alpha$. This kind of syntax is used in epistemic logic [34,32], but the usual semantics in terms of accessibility relations, often motivated by the modeling of introspection, does not easily fit with uncertainty formalisms like probability or possibility theories, that rely on weights assigned to possible worlds. Kripke semantics are better tailored for other applications such as temporal logic.

The aim of this paper is to define a minimal language that makes it possible to reason about partial information provided by a logically sophisticated agent. A set of formulae in this language represents what an agent can sincerely reveal about his or her knowledge or beliefs. An agent is viewed here as a source of information, or a witness. In this language, atomic propositions are expressed as $\Box\alpha$, where α is any formula from a propositional language and $\Box$, expressing belief

* Corresponding author at: IRIT, CNRS Université de Toulouse, France.

0888-613X/$ – see front matter
http://dx.doi.org/10.1016/j.ijar.2013.11.003

or knowledge,[1] is borrowed from modal logics. The language is then completed by means of classical negation and conjunction. However, the nesting of modalities is not allowed because we are not concerned with introspective nor multiagent reasoning. The obtained language is an elementary fragment of well-known modal systems. This "minimal epistemic logic" can also be understood as a *meta-epistemic* logic (MEL) since we take an imperfect external point of view on the agent knowledge, in the sense of Aucher [1].

At the semantic level, we use the simplest basic representation of incomplete information common to all uncertainty theories. Incomplete knowledge about the real world possessed by an agent will be represented just by a non-empty subset of interpretations, one and only one of which is, according to this agent's beliefs,[2] the actual state of the world. This is what is usually called an *epistemic state*. Moreover, all that is known about the agent's epistemic state stems from what this agent sincerely reported. So we have incomplete knowledge about this epistemic state (we call it a *meta-epistemic state*). This kind of representation of higher order incomplete knowledge already exists in uncertainty theories. In Shafer's theory of evidence [47], a belief function is represented by a probability distribution over epistemic states.

The semantics of the proposed logic is in terms of epistemic states. It does not use full-fledged Kripke-style semantics nor does it evaluate modal formulae on propositional valuations. In this sense MEL, even if syntactically a fragment of a known modal logic, is not really in the spirit of the standard modal logic trend for representing and reasoning about knowledge.

The originality of the paper lies in its concern for a minimal language for reasoning about incomplete information revealed by an agent, the non-Kripke semantics of the system MEL, and the connection between modal logic and uncertainty theories it suggests. In fact, such a link is briefly outlined in a book of Hájek [28] who developed it with colleagues for probability theory [29], gradual possibility theory [30] or belief functions [27] using formal fuzzy logics. The paper considers the simplest Boolean core of such logics, so as to highlight its central role in logics of uncertainty.

The paper[3] is organized as follows. In the next section, the syntax and the axiomatic setting of the logic MEL are provided. The set-valued semantics of the logic is then supplied in Section 3; soundness and a proof of completeness with respect to the intended semantics is established, that relies only on the completeness of propositional logic, and the use of possibility theory. We explain how to encode any set of epistemic models as a MEL-formula. We also show in which sense MEL is a two-tiered propositional logic, rather than a usual modal logic. The relationship to uncertainty theories, like probability, possibility and Shafer's theory of evidence is described in Section 4. It is shown that there is a MEL-formula encoding a single epistemic state, that is the logical counterpart to the Möbius transform of a belief function. The latter can then be viewed as a probabilistic rendering of a meta-epistemic state. In Section 5, some related works are discussed further. Finally, perspectives are outlined in Section 6.

2. The logic MEL: syntax and axioms

The language of the proposed logic aims at enabling an agent to sincerely provide some information about his or her beliefs on the outside world, so as to enable another agent to reason about it. As hinted in the introduction, in this paper, we interchangeably use the words knowledge and belief, as the formalism is too elementary to make the distinction. Note that while this is in opposition to the philosophical tradition, it is in line with Artificial Intelligence, where the terms "knowledge base" and "belief base" are often used indifferently. Moreover while the philosophic tradition often interprets knowledge as true belief, other authors [41] consider belief as defeasible knowledge. MEL is essentially a logic for reasoning under uncertainty based on an agent's revealed incomplete information about the world. Whether this information is in conformity with the real world or not is not the point here. Moreover, we exclude introspection from our concerns. The proposed syntax makes it possible to express whether a classical proposition is believed, or unknown to the agent. The information possessed by the agent is said to be incomplete, when the truth of some proposition remains unknown for this agent.

2.1. Syntax

Let us consider classical propositional logic PL, with (say) k propositional variables, $p_1, \ldots, p_k$. Let $\alpha, \beta, \ldots$ denote PL-formulae obtained as usual by Boolean connectives $\neg, \wedge$, forming the language $\mathcal{L}$. The main idea of the proposed syntax is to *encapsulate* PL inside a language equipped with a modality denoted by $\Box$. This is in contrast with usual epistemic modal logics that *extend* PL with this additional symbol. The intended purpose here is to completely separate propositions in $\mathcal{L}$ referring to the real world and propositions that refer to an agent's epistemic state, where the symbol $\Box$ appears.

We thus construct atoms of MEL by adding the unary connective $\Box$ in front of all sentences in $\mathcal{L}$ – so atomic formulae of MEL are of the form $\Box\alpha$, $\alpha \in \mathcal{L}$, and form the set At. The intended meaning of $\Box\alpha$ is that an agent knows (or believes) proposition α is true, that is, α holds in *every* possible world compatible with this agent's epistemic state.

[1] In this elementary language, the distinction between belief and knowledge cannot be made.

[2] We do not make any presupposition as to whether the beliefs are warranted or not.

[3] A short preliminary version [2] of this paper was presented at the 10th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, Verona, July 2009, and at the Dagstuhl Seminar on "Information processing, rational belief change and social interaction", August 2009.