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Real-valued Choquet integrals for set-valued mappings ☆

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ABSTRACT

In this paper a new kind of real-valued Choquet integrals for set-valued mappings is introduced, and some elementary properties of this kind of Choquet integrals are studied. Convergence theorems of a sequence of Choquet integrals for set-valued mappings are shown. However, in the case of the monotone convergence theorem of the nonincreasing sequence of Choquet integrals for set-valued mappings, we point out that the integrands must be closed. Specially, this kind of real-valued Choquet integrals for set-valued mappings can be regarded as the Choquet integrals for single-valued functions.

1. Introduction

The Choquet integral with respect to a fuzzy measure was proposed by Murofushi and Sugeno [1]. It was introduced by Choquet [2] in potential theory with the concept of capacity. Then, it has been used for utility theory in the field of economic theory [3], and has been used for image processing, pattern recognition, information fusion and data mining [4–7], in the context of fuzzy measure theory [8–12]. But all the integrands in these papers are single-valued functions, and the Choquet integral of a nonnegative single-valued function is defined as

$$(c)\int_A f\,d\mu = \int_0^\infty \mu(f_\alpha \cap A)\,d\alpha,$$

where f is a nonnegative measurable single-valued function, $f_\alpha = \{x \in X \mid f(x) \geqslant \alpha\}$.

It is well known that set-valued mappings have been used repeatedly in economics [13]. Integrals of set-valued mappings had been studied by Aumann [14]. By using the approach of Aumann, Jang et al. [16,17] defined Choquet integrals of set-valued mappings as

$$(c)\int_A F\,d\mu = \left\{(c)\int_A f\,d\mu \;\middle|\; f \in S(F)\right\}, \tag{1}$$

where F is a measurable set-valued mapping, $S(F)$ denotes the family of Choquet measurable selection of F.

In this paper we introduce another kind of Choquet integrals for set-valued mappings in similar form as fuzzy integrals for set-valued mappings in [15] as follows:

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$$(c)\int_A F\,d\mu=\int_0^\infty \mu(F_\alpha\cap A)\,d\alpha, \tag{2}$$

where F is a measurable set-valued mapping, $F_\alpha=\{x\in X \mid F(x)\cap[\alpha,\infty]\neq\phi\}$. Specially, the kind of Choquet integral is equal to the Choquet integral for a single-valued function, namely,

$$(c)\int_A F\,d\mu=(c)\int_A f\,d\mu,$$

where $f(x)=\sup F(x)=\sup\{y \mid y\in F(x)\}$ for every $x\in X$ (Theorem 1(2)). Obviously, the Choquet integral in Eq. (1) is set-valued but our Choquet integral in Eq. (2) is real-valued. Moreover, our Choquet integral is not the special case of the one in [16–18] and the discussion manner is also quite different.

This paper is organized as follows. Section 2 presents some concepts on fuzzy measures and set-valued mappings. Section 3 defines the real-valued Choquet integrals for set-valued mappings and shows the basic properties. Section 4 investigates the convergence of a sequence of Choquet integrals for set-valued mappings and obtains some results as follows:

(1) μ is continuous from below $\Longleftrightarrow$ for any $F_n\uparrow F$ we have $(c)\int F_n\,d\mu\uparrow(c)\int F\,d\mu$;
(2) μ is conditionally continuous from above $\Longleftrightarrow$ for any $F_n\downarrow F$ with $(c)\int F_{n_0}\,d\mu<\infty$ for some $n_0\in N$ we have $(c)\int F_n\,d\mu\downarrow(c)\int F\,d\mu$ (F_n, F is closed);
(3) $\mu_n\uparrow\mu \Longleftrightarrow$ for any F we have $(c)\int F\,d\mu_n\uparrow(c)\int F\,d\mu$;
(4) $\mu_n\downarrow\mu \Longleftrightarrow$ for any F with $(c)\int F\,d\mu_{n_0}<\infty$ for some $n_0\in N$ we have $(c)\int F\,d\mu_n\downarrow(c)\int F\,d\mu$.

Section 5 concludes this paper.

2. Preliminaries

In the paper the following concepts and notations will be used. $R^+=[0,\infty]$ denotes the set of extended nonnegative real numbers. $\mathcal{P}(R^+)$ denotes the class of all the subsets of R^+, $\mathcal{C}(R^+)$ denotes the class of all the closed subsets of R^+. X denotes a nonempty set, $\mathcal{A}$ is a σ-algebra on X, and $(X,\mathcal{A})$ is a measurable space.

Let $A\subset R^+$, if A is bounded from above, define $\sup A=$ the least upper bound of A; if A is unbounded from above, define $\sup A=\infty$. Hence for every $A\subset R^+$, $\sup A$ is always well-defined.

Definition 1. (See [9].) Let $\mu:\mathcal{A}\to[0,\infty]$ be a set function. μ is called a fuzzy measure if it satisfies the following conditions:

(1) $\mu(\phi)=0$;
(2) $\mu(A)\leqslant\mu(B)$ whenever $A\subset B$, $A,B\in\mathcal{A}$.

Definition 2. (See [9].) Let $\mu:\mathcal{A}\to[0,\infty]$ be a fuzzy measure.

(1) μ is said to be continuous from below if $A_n\subset A_{n+1}$, $A_n\in\mathcal{A}$, $n\in N$, then $\mu(\cup_{n=1}^\infty A_n)=\lim_n\mu(A_n)$;
(2) μ is said to be conditionally continuous from above if $A_n\supset A_{n+1}$, $A_n\in\mathcal{A}$, $n\in N$ and $\mu(A_{n_0})<\infty$ for some $n_0\in N$, then $\mu(\cap_{n=1}^\infty A_n)=\lim_n\mu(A_n)$.

A set-valued mapping is a mapping $F:X\to\mathcal{P}(R^+)\setminus\{\phi\}$, and it is said to be measurable if

$$F^{-1}(B)=\big\{x\in X \bigm| F(x)\cap B\neq\phi\big\}\in\mathcal{A}$$

for every $B\in\mathcal{B}(R^+)$, where $\mathcal{B}(R^+)$ is the Borel algebra of R^+.

Definition 3. Let $F,G:X\to\mathcal{P}(R^+)\setminus\{\phi\}$ be measurable set-valued mappings and μ a fuzzy measure on $(X,\mathcal{A})$. If $\mu(\{x \mid F(x)\neq G(x)\})=0$, then we say F equals G almost everywhere, denoted by $F=G$ a.e.

3. Real-valued Choquet integrals for set-valued mappings

When μ is a fuzzy measure, the triple $(X,\mathcal{A},\mu)$ is called a fuzzy measure space. Throughout this paper, unless otherwise stated, the following are discussed on the fuzzy measure space $(X,\mathcal{A},\mu)$.

Definition 4. Let $F:X\to\mathcal{P}(R^+)\setminus\{\phi\}$ be a measurable set-valued mapping and $A\in\mathcal{A}$. Then the real-valued Choquet integral of F on A is defined as