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Membership function based rough set

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ABSTRACT

In this paper the notion of a kind of clusters of subsets of a set based on rough membership function is introduced. The algebraic structure emerged thereby is studied. A comparison with classical rough sets with respect to the algebraic properties has been made. A many-valued propositional logic for such entities is proposed. Representation theorems in the style of Obtułowicz have been established.

1. Introduction

This paper continues a series of works connecting rough sets and fuzzy sets following and developing the method adopted in Obtułowicz's paper [9], the first publication in this direction being in 2011 [2]. Since the inception of rough set theory by Pawlak in 1982 [13], researchers had been interested in the interrelation between this theory and the theory of fuzzy sets [25] that was already prevailing the scenario. Some instances of this group of research are mentioned in the references [2,4,5,9,14,17,21–23].

In the present paper a notion of rough membership function based rough set or simply, membership function based rough set (henceforth *MF*-rough set) is introduced. It will be apparent that an *MF*-rough set viewed as rough membership function may be considered as a fuzzy set. The algebra of *MF*-rough sets is developed. This algebra has been compared with the rough set algebra defined in [1]. A logic will be proposed which is endowed with *MF*-rough set semantics. Then representation theorems are proved in the line of Obtułowicz linking *MF*-rough sets and a kind of fuzzy binary relations having a special linear lattice of rational numbers as value set. The main departure from Obtułowicz however lies in not taking the value set a Heyting algebra. The reason for this deviation will be clarified later at the appropriate place. This paper is an improvement in a sense over the earlier work [2]. The entire study of the algebraic structure of rough membership functions is a new addition. Besides, the logic developed here is propositional but in [2] it was a predicate logic and quite different in its propositional content. Thirdly the formalism in connection with Obtułowicz-like representation of rough sets is also an improvement [cf. the note at the end of Section 5].

As presented by Pawlak, at the origin of rough set theory lies a set X of objects and an equivalence relation R generated by an attribute-value system. The pair $\langle X, R\rangle$ is called an approximation space. The equivalence class $[\cdot]_R$ will also be called a block of R. For any subset $A \subseteq X$, two approximations viz. lower and upper are defined by $\underline{A} = \{x \mid [x]_R \subseteq A\}$ and $\overline{A} = \{x \mid [x]_R \cap A \neq \phi\}$.

These approximations of the set A are at the root of the theory. A concept with the extension A is approximated by two rather well defined concepts with extensions $\underline{A}$ and $\overline{A}$. Subsequent years have witnessed various methods of defining these approximations based on various practical as well as theoretical motivations [6,10–12,15,19,20]. The above approximation

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based rough set theory is now called classical or Pawlakian. But even within the classical theory there are some differences in the exact definition of what a rough set is. Depending upon various points of view and usages there may be various definitions (see [1]). In this paper we take the following one.

Definition 1.1. A rough set is a triple $\langle X, R, [\cdot]_{\approx}\rangle$ where X is a non-empty set, R is an equivalence relation on X and $[\cdot]_{\approx}$ is an equivalence class with respect to the relation $\approx$ of rough equality on the power set $\wp(X)$ of X viz. $A \approx B$ if and only if $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$, $A, B \subseteq X$.

The most popular definition is however given by the pair $\langle \underline{A}, \overline{A}\rangle$. Neither of the two definitions of a rough set requires finiteness of the universe X nor any restriction on R but in most of the applications, as is well understood, the universe is taken as finite.

This paper is organized as below.

In Section 2 rough membership function based rough set is defined and a few basic theorems are established. Section 3 deals with the algebra of rough membership functions vis-à-vis these newly defined entities. In Section 4 a many-valued propositional logic has been proposed and a semantics is given interpreting a wff of the logic as an *MF*-rough set. Representation theorems in the style of Obtułowicz is presented in Section 5. Section 6 contains some concluding remarks.

2. *MF*-rough sets

Taking the universe X as finite the notion of rough membership function was formally defined by Pawlak and Skowron in [16] and applied to develop rough mereology [17,18].

Definition 2.1. Given any subset $A \subseteq X$, a rough membership function f_A is a mapping from X to $Ra[0,1]$, the set of rational numbers in $[0,1]$, defined by $f_A(x) = \frac{Card([x]_R \cap A)}{Card([x]_R)}$ for all $x \in X$.

A basic assumption. For our purpose, we take X as any set, finite or infinite, but assume that the equivalence classes $[\cdot]_R$ or blocks generated by R are all of finite cardinality.

Observation 2.2. $f_A(x) = 1$ *if and only if* $x \in \underline{A}$.
$f_A(x) = 0$ *if and only if* $x \in (\overline{A})^c$.
$0 < f_A(x) < 1$ *if and only if* $x \in Bd(A) = \overline{A} - \underline{A}$.
$f_A(x) = f_A(y)$ *for* xRy.
If $[\cdot]_R \subseteq Bd(A)$, $[\cdot]_R$ *is not a singleton.*

Observation 2.3. *Each block* $[\cdot]_R$ *being finite, there is a fixed set of rational numbers in* $[0,1]$ *that are admissible values for the members of the block viz.* $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$, *where* $Card([\cdot]) = n$.

This set of admissible values is determined right at the beginning when the partition is formed in X. Under a rough membership function f_A *all elements of a block receive the same value out of the set of admissible values associated with the particular block which will be denoted by admiss-value*$[\cdot]$. *This value shall also be referred to as the value of the block under the rough membership function and denoted by* $f_A([\cdot])$.

Observation 2.4. *Some properties of rough membership functions are listed below.*

(i) *If* $f_A = f_B$ *then* $A \approx B$ *but the converse does not hold.*
(ii) *If* $A \approx B$ *then* $f_A(x) = 1$ *if and only if* $f_B(x) = 1$ *and* $f_A(x) = 0$ *if and only if* $f_B(x) = 0$.
(iii) *If for some* $A, x, 0 < f_A(x) < 1$ *then there exists* $B \neq A$ *such that* $f_A = f_B$.
(iv) $f_{A^c}(x) = 1 - f_A(x)$ *for all* $x \in X$.
(v) *If* $A \subseteq B$ *then* $f_A \leqslant f_B$, *but the converse does not hold.*
(vi) *If* $f_A \leqslant f_B$ *then* $\underline{A} \subseteq \underline{B}$ *and* $\overline{A} \subseteq \overline{B}$ *i.e. A is roughly included in B.*
(vii) $\max[0, f_A(x) + f_B(x) - 1] \leqslant f_{A\cap B}(x) \leqslant \min[f_A(x), f_B(x)]$.
(viii) $\max[f_A(x), f_B(x)] \leqslant f_{A\cup B}(x) \leqslant \min[1, f_A(x) + f_B(x)]$.
(ix) $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_{A\cap B}(x)$.

The results (vii), (viii) and (ix) are proved by Yao [23].

We now give the definition of an *MF*-rough set.

Definition 2.5. Let $\equiv$ be the relation defined on $\wp(X)$ by $A \equiv B$ if and only if $f_A = f_B$. $\equiv$ is an equivalence relation generating a partition on $\wp(X)$.

An *MF*-rough set is a triple $\langle X, R, [\cdot]_{\equiv}\rangle$ where X, R are as before and $[\cdot]_{\equiv}$ is a member of the quotient set $\wp(X)/_{\equiv}$.