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The concept lattice functors ☆

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ABSTRACT

This paper is concerned with the relationship between contexts, closure spaces, and complete lattices. It is shown that, for a unital quantale L, both formal concept lattices and property oriented concept lattices are functorial from the category L-**Ctx** of L-contexts and infomorphisms to the category L-**Sup** of complete L-lattices and suprema-preserving maps. Moreover, the formal concept lattice functor can be written as the composition of a right adjoint functor from L-**Ctx** to the category L-**Cls** of L-closure spaces and continuous functions and a left adjoint functor from L-**Cls** to L-**Sup**.

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1. Introduction

A *formal context* is a triple (X, Y, R), where X, Y are sets and $R \subseteq X \times Y$ is a relation from X to Y. In a formal context (X, Y, R), X is considered to be the set of *objects*, Y the set of *properties*, and $(x, y) \in R$ means that the object x has the property y. Formal contexts provide a common framework for formal concept analysis (FCA) [7,10] and rough set theory (RST) [11,26]. Given a context (X, Y, R), there exists a contravariant Galois connection $(R_\uparrow, R^\downarrow)$ and a covariant Galois connection $(R_\exists, R^\forall)$ between the powersets of X and Y. These two Galois connections play fundamental roles in formal concept analysis and rough set theory respectively.

A *formal concept* [10] of the context (X, Y, R) is a pair $(U, V) \in 2^X \times 2^Y$ satisfying $U = R^\downarrow(V)$ and $V = R_\uparrow(U)$; a *property oriented concept* [11,26,27] of the context (X, Y, R) is a pair $(U, V) \in 2^X \times 2^Y$ satisfying $U = R^\forall(V)$ and $V = R_\exists(U)$. The set of all the formal concepts of the context (X, Y, R) is denoted by $\mathfrak{B}(X, Y, R)$, and the set of all the property oriented concepts by $\mathfrak{P}(X, Y, R)$. Both $\mathfrak{B}(X, Y, R)$ and $\mathfrak{P}(X, Y, R)$ are complete lattices.

This paper is concerned with the functorial properties of $\mathfrak{B}$ and $\mathfrak{P}$ (see [19] for category theory). To this end, we must determine the morphisms between contexts and that between complete lattices. There are different approaches to morphisms between contexts, see e.g., [8–10,15,16,20,25,28]. [1] In particular, Mori [20] has shown that the construction of formal concept lattices induces an equivalence between the category of contexts and Chu correspondences and that of complete lattices and suprema-preserving maps.

In this paper, we consider infomorphisms between formal contexts. It is shown that both the construction of formal concept lattices and that of property oriented concept lattices are functorial from the category **Ctx** of contexts and

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[1] In rough set theory, there exist different approaches to morphisms between formal contexts of the form (X, X, R) with a subset $A \subseteq X$, where R is an equivalence relation on X, see Banerjee and Chakraborty [1], Banerjee and Yao [2] for instance. We are grateful to one of the reviewers for bringing [1,2] to our attention.

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infomorphisms to the category **Sup** of complete lattices and suprema-preserving maps. It should be noted that an infomorphism is not a Chu correspondence in the sense of Mori [20] in general, hence the functoriality discussed here does not follow from the result of Mori.

An infomorphism [9,16] $(f, g) : (X, Y, R) \longrightarrow (A, B, S)$ between contexts is a pair of functions $f : X \longrightarrow A$ and $g : B \longrightarrow Y$ such that $(x, g(b)) \in R$ if and only if $(f(x), b) \in S$ for all $x \in X$ and $b \in B$. Contexts and infomorphisms constitute a category **Ctx**.

Though it is possible to show that $\mathfrak{B}, \mathfrak{P}$ are both functorial from **Ctx** to **Sup** in a direct way, we will establish the functoriality of $\mathfrak{B}, \mathfrak{P}$ by help of closure spaces. The benefit of doing so is that we will obtain decompositions of the functors $\mathfrak{B}$ and $\mathfrak{P}$, these decompositions are helpful for further investigation on these functors and the interrelationship between contexts, closure spaces, and complete lattices.

Let X be a set, a closure operator is an order-preserving map $c : 2^X \longrightarrow 2^X$ on the powerset of X such that $A \subseteq c(A)$ for all $A \subseteq X$ and $c \circ c = c$. The pair (X, c) is called a closure space, a subset $A \subseteq X$ is closed if $A = c(A)$. A map $f : (X, c) \longrightarrow (Y, d)$ between closure spaces is continuous if $f(c(A)) \subseteq d(f(A))$ for each subset A of X. The category of closure spaces and continuous maps is denoted by **Cls**.

Given a context (X, Y, R), both $R^{\downarrow} \circ R_{\uparrow}$ and $R^{\forall} \circ R_{\exists}$ are closure operators on X. The correspondence $(X, Y, R) \mapsto (X, R^{\downarrow} \circ R_{\uparrow})$ defines a functor U : **Ctx** $\longrightarrow$ **Cls** which has a left adjoint.

Given a closure space (X, c), the set $c(2^X)$ of all the closed subsets of (X, c) is a complete lattice. The correspondence $(X, c) \mapsto c(2^X)$ defines a functor T : **Cls** $\longrightarrow$ **Sup** which has a right adjoint.

For each context (X, Y, R), $T \circ U(X, Y, R)$ is isomorphic to the formal concept lattice $\mathfrak{B}(X, Y, R)$. Hence $\mathfrak{B}$ is functorial from **Ctx** to **Sup**, and it is the composition of a right adjoint functor U : **Ctx** $\longrightarrow$ **Cls** and a left adjoint functor T : **Cls** $\longrightarrow$ **Sup**.

The property oriented concept lattice functor can also be written as a composition of a functor V : **Ctx** $\longrightarrow$ **Cls** and the functor T : **Cls** $\longrightarrow$ **Sup**.

All the conclusions stated above will be proved in a much more general setting in this paper. The theories of formal concept lattices and property oriented concept lattices have been generalized to the fuzzy setting [6,11–13,17,21,26]. We shall prove the L-version of the conclusions stated above for a unital quantale $(L, \&)$.

The contents are arranged as follows. Section 2 recalls some basic notions of quantales. Section 3 introduces the categories considered in this paper: the category L-**Ctx** of L-contexts and infomorphisms, the category L-**Sup** of complete L-lattices and suprema-preserving maps, and the category L-**Cls** of L-closure spaces and continuous maps. An adjunction between L-**Cls** and L-**Sup** is given in Section 4. Section 5 presents an adjunction between L-**Ctx** and L-**Cls**. In Section 6, it is demonstrated that both the formal concept lattices and property oriented concept lattices of L-contexts are functorial from L-**Ctx** to L-**Sup**, and the formal concept lattice functor is the composition of a right adjoint functor and a left adjoint functor. And, if $(L, \&)$ has a dualizing element, then the property oriented concept lattice functor is the composition of the formal concept lattice functor following a functor L-**Ctx** $\longrightarrow$ L-**Ctx** and vice versa.

2. Quantales

A *quantale* [22] is a pair $(L, \&)$, where L is a complete lattice, $\&$ is an associative binary operation on L such that $a\&(\bigvee b_i) = \bigvee(a\&b_i)$ and $(\bigvee b_i)\&a = \bigvee(b_i\&a)$ for all $a, b_i \in L$. The top and the bottom element of L is denoted by 1 and 0 respectively. A quantale $(L, \&)$ is said to be *unital* if there exists an element $I \in L$ such that $I\&a = a = a\&I$ for all $a \in L$. Finally, $(L, \&)$ is commutative if $a\&b = b\&a$ for all $a, b \in L$.

Definition 2.1 [22]. Let $(L, \&)$ be a quantale. Define $\swarrow, \searrow : L \times L \longrightarrow L$ by

$$c \swarrow b = \bigvee\{a \in L : a\&b \le c\} \text{ and } b \searrow c = \bigvee\{a \in L : b\&a \le c\}.$$

If $(L, \&)$ is commutative, then $c \swarrow b = b \searrow c$ for all $b, c \in L$ and will be denoted by $b \to c$.

Proposition 2.2 [22]. *Let $(L, \&)$ be a quantale. The following properties hold for all $a, b, c, a_t, b_t \in L$:*

(1) $a \le c \swarrow b \iff a\&b \le c \iff b \le a \searrow c$.
(2) $(\bigwedge b_t) \swarrow a = \bigwedge(b_t \swarrow a)$; $a \searrow (\bigwedge b_t) = \bigwedge(a \searrow b_t)$.
(3) $b \swarrow (\bigvee a_t) = \bigwedge(b \swarrow a_t)$; $(\bigvee a_t) \searrow b = \bigwedge(a_t \searrow b)$.
(4) $(a \searrow b)\&(b \searrow c) \le a \searrow c$; $(c \swarrow b)\&(b \swarrow a) \le c \swarrow a$.
(5) $(c \swarrow b) \swarrow a = c \swarrow (a\&b)$; $a \searrow (b \searrow c) = (b\&a) \searrow c$.
(6) $a \searrow (c \swarrow b) = (a \searrow c) \swarrow b$.
(7) $a\&(a \searrow b) \le b$; $(b \swarrow a)\&a \le b$.

An element d in a quantale $(L, \&)$ is *cyclic* [22] if $d \swarrow a = a \searrow d$ for all $a \in L$. In this case, we write $a \to d$ for $d \swarrow a = a \searrow d$. It is easy to check that a quantale $(L, \&)$ is commutative if and only if every element of L is cyclic. An element d in $(L, \&)$ is *dualizing* [22] if $d \swarrow (a \searrow d) = a = (d \swarrow a) \searrow d$ for all $a \in L$.

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