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Block relations in formal fuzzy concept analysis

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ABSTRACT

One of the main problems in formal concept analysis (especially in fuzzy setting) is to reduce a concept lattice of a formal context to appropriate size to make it graspable and understandable. A natural way to do it is to substitute the formal context by its block relation which is equivalent to factorization of the concept lattice by a complete tolerance. We generalize known results on the correspondence of block relations of formal contexts and complete tolerances on concept lattices to fuzzy setting and we provide an illustrative example of using block relations to reduce the size of a concept lattice.

1. Introduction

The present paper studies block relations and related structures in formal fuzzy concept analysis.

Formal concept analysis (FCA) [16,14] is a method of exploratory data analysis based on a formalization of a philosophical view of conceptual knowledge. The basic notion in FCA is that of a formal concept which consists of two sets: extent – a set of all objects sharing the same attributes, and intent – a set of all the shared attributes. This definition of formal concept comes from traditional (Port-Royal) logic [1,22]. The input data for FCA (in basic setting), called a formal context, is a flat table in which rows represent objects and columns represent attributes. Entries of the table contain 1 (or $\times$), which means that the corresponding object has the corresponding attribute, or 0 (blank) which means the opposite. The main output is a hierarchy of formal concepts in the table.

In everyday life we use concepts which are not sharply bounded (e.g. 'great dancer' or 'middle aged man'). In terms of FCA, the formal concepts do not divide objects and attributes sharply into those which are covered and which are not; it is rather a matter of degree. There are several approaches to generalize formal concept analysis to work with graded data [8,5,29,27,21,13]. Many of them are based on the Zadeh's theory of fuzzy sets [33]. Our work follows approach of [8].

One of the main problems in FCA (especially in fuzzy setting) is to reduce the size of a concept lattice to make it graspable and understandable. One method to achieve it is to use a block relation and obtain rougher data which contain a smaller number of formal concepts. That (in the crisp case) corresponds to particular factorization of the associated concept lattice [32,16], or to particular automorphism of the concept lattice. We generalize these known results to fuzzy setting.

In [26] we have studied a generalization of bonds—intercontextual structures binding two fuzzy contexts—and related morphisms of associated concept lattices. A few specific cases of these structures (in the crisp case) deserve a special attention at their generalization to fuzzy setting. For instance, infomorphisms, scale measures, and presently studied block relations are such cases. We study block relations in fuzzy setting for two main reasons. First, in the crisp setting they

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correspond to another interesting notion—complete tolerances on associated concept lattices [32,16]. Second, they provide a natural way to reduce a concept lattice [28].

The paper is structured as follows. Section 2 recalls notions used in the paper. In Sections 3.1–3.3 we separately study three instances of block **L**-relations. In addition, Sections 2.4, 3.1, 3.3, and 3.5 contain a central running example of this paper.

2. Preliminaries

2.1. Residuated lattices, fuzzy sets, and fuzzy relations

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\to$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \to c$.

0 and 1 denote the least and greatest element, respectively. The partial order of **L** is denoted by $\leq$. Throughout this work, **L** denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\to$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \to 0 \tag{1}$$

An **L**-set A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$ [20,19]. The set of all **L**-sets in a universe X is denoted L^X.

Operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A, B \in L^X$ is an **L**-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc.

Intersection and union of two **L**-sets can be generalized to any number of **L**-sets and even to **L**-sets of **L**-sets. For an **L**-set $U : L^X \to L$, the intersection $\bigcap U$ and union $\bigcup U$ of U are **L**-sets in X, defined by

$$\bigcap U(x) = \bigwedge_{A \in L^X} U(A) \to A(x), \qquad \bigcup U(x) = \bigvee_{A \in L^X} U(A) \otimes A(x), \tag{2}$$

for any $x \in X$.

We often use the following notation to specify fuzzy sets. If $x_1, x_2, \ldots, x_n \in X$ are pairwise distinct and $a_1, a_2, \ldots, a_n \in L$ then $\{{}^{a_1}/x_1, {}^{a_2}/x_2, \ldots, {}^{a_n}/x_n\}$ denotes the **L**-set A given by $A(x) = a_k$ if $x = x_k$ for some $k \in \{1, 2, \ldots, n\}$ and $A(x) = 0$ otherwise. More generally, for an index set K, let for each $k \in K$, $a_k \in L$ and $x_k \in X$ be pairwise distinct. We denote by $\{{}^{a_k}/x_k \mid k \in K\}$ the **L**-set A satisfying $A(x) = a_k$ if $x = x_k$ for some $k \in K$ and $A(x) = 0$ otherwise.

Sometimes, it is useful to allow repeated occurrences of elements of X in this notation. In this case the membership degree of each element is obtained as supremum of all its listed degrees: if $A = \{{}^{a_1}/x_1, {}^{a_2}/x_2, \ldots, {}^{a_n}/x_n\}$ then

$$A(x) = \bigvee \{a_k \mid k \in \{1, \ldots, n\} \text{ and } x_k = x\}$$

and if $A = \{{}^{a_k}/x_k \mid k \in K\}$ then

$$A(x) = \bigvee \{a_k \mid k \in K \text{ and } x_k = x\}.$$

An **L**-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in L^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, the **L**-sets $a \otimes A \in L^X$, $a \to A$, $A \to a$, and $\neg A$ in X are defined by

$$(a \otimes A)(x) = a \otimes A(x), \tag{3}$$

$$(a \to A)(x) = a \to A(x), \tag{4}$$

$$(A \to a)(x) = A(x) \to a, \tag{5}$$

$$\neg A(x) = A(x) \to 0. \tag{6}$$

For $A \in L^X$ the **L**-sets $a \otimes A, a \to A, A \to a$ are called a-multiplication, a-shift, and a-complement, respectively. By (2) we have $a \otimes A = \bigcup \{{}^a/A\}$ and $a \to A = \bigcap \{{}^a/A\}$.