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Representing qualitative capacities as families of possibility measures

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ABSTRACT

This paper studies the structure of qualitative capacities, that is, monotonic set-functions, when they range on a finite totally ordered scale equipped with an order-reversing map. These set-functions correspond to general representations of uncertainty, as well as importance levels of groups of criteria in multiple-criteria decision-making. We show that any capacity or fuzzy measure ranging on a qualitative scale can be viewed both as the lower bound of a set of possibility measures and the upper bound of a set of necessity measures (a situation somewhat similar to the one of quantitative capacities with respect to imprecise probability theory). We show that any capacity is characterized by a non-empty class of possibility measures having the structure of an upper semi-lattice. The lower bounds of this class are enough to reconstruct the capacity, and the number of them is characteristic of its complexity. An algorithm is provided to compute the minimal set of possibility measures dominating a given capacity. This algorithm relies on the representation of the capacity by means of its qualitative Möbius transform, and the use of selection functions of the corresponding focal sets. We provide the connection between Sugeno integrals and lower possibility measures. We introduce a sequence of axioms generalizing the maxitivity property of possibility measures, and related to the number of possibility measures needed for this reconstruction. In the Boolean case, capacities are closely related to non-regular modal logics and their neighborhood semantics can be described in terms of qualitative Möbius transforms.

1. Introduction

A fuzzy measure [38] (or a capacity [10]) is a set-function that is monotonic under inclusion. In this paper, the capacity is said to be *qualitative* (or *q-capacity*, for short) if its range is a finite totally ordered set. It means we do not presuppose addition is available in the capacity range, only minimum and maximum. In such a context the connection with probability measures is lost. Consequently a number of notions, meaningful in the numerical setting, are lost as well, such as the Möbius transform [34], the conjugate, supermodularity [10] and the like. Likewise, some numerical capacities (such as convex ones, belief functions) can be viewed as encoding a convex family of probability distributions [35,40]. This connection disappears if we give up using addition in the range of the capacity.

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Yet, it is tempting to check whether counterparts of many such quantitative notions can be defined for qualitative capacities, if we replace probability measures by possibility measures. Conjugateness can be recovered if the range of the capacity is equipped with an order-reversing map. A qualitative counterpart of a Möbius transform has been introduced by Mesiar [31] and Grabisch [25] in 1997 and further studied by Grabisch [26]. The qualitative Möbius transform can be viewed as the possibilistic counterpart to a basic probability assignment, whereby a capacity is defined with respect to the latter by a qualitative counterpart of the belief function definition, extending as well the definition of possibility measures. In fact the process of generation of belief functions, introduced by Dempster [9], was applied very early to possibility measures by Dubois and Prade [16,17] so as to generate upper and lower possibilities and necessities. It was noticed that upper possibilities and lower necessities are still possibility and necessity measures respectively, but upper necessities and lower possibilities are not. However, as we shall see, the formal analogy between belief functions and qualitative capacities via the qualitative Möbius transform can be misleading at the interpretive level.

This situation leads to natural questions, namely, whether a qualitative capacity can be expressed in terms of a family of possibility measures, and if a qualitative Möbius transform can encode such a family. Previous recent works [11,32] started addressing this issue, taking up a pioneering work by Banon [4]. In this paper we show that in the finite (qualitative) setting, special subsets of possibility measures play a role similar to convex sets of probability measures. We prove that any capacity can be defined in terms of a finite set of possibility measures, either as a lower possibility or an upper necessity function. This result should not come as a surprise. Indeed, it has been shown that possibility measures can be refined by probability measures using a lexicographic refinement of the basic axiom of possibility measures, and that capacities on a finite set can be refined by belief functions [13,14]. Based on this fundamental result we can generalize the maxitivity and minitivity axiom of possibility theory so as to define families of qualitative capacities of increasing complexity. Finally, this property enables qualitative capacities to be seen as necessity modalities in a non-regular class of modal logics, extending the links between possibility theory and modal logic to a potentially larger range of uncertainty theories.

The structure of the paper is as follows.[1] Section 2 provides basic definitions pertaining to capacities, recalls and discusses the similarity between belief functions and qualitative capacities, indicating the limitation of this analogy. Section 3 provides the main contribution of this paper, namely it shows the formal analogy between qualitative capacities and imprecise probabilities, proving any capacity comes down to any of two families of possibility distributions, and can be described by finite sets thereof, either as a lower possibility or an upper necessity function. This section also extends these results to Sugeno integrals. Section 4 provides an algorithm that computes the set of minimal elements among possibility measures that dominate a capacity from its qualitative Möbius transforms. Section 5 axiomatically defines subfamilies of qualitative capacities of increasing complexity generalizing the maxitive and the minitivity axioms of possibility theory. Finally, in Section 6 we lay bare a connection between capacities and neighborhood semantics in non-regular modal logics, which suggests potential applications to reasoning from conflicting information coming from several sources.

2. Qualitative capacities and Möbius transforms

Consider a finite set S and a finite totally ordered scale $L = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_\ell = 1\}$ with top 1 and bottom 0. Moreover we assume that L is equipped with an order-reversing map, i.e., a strictly decreasing mapping $\nu : L \to L$ with $\nu(1) = 0$ and $\nu(0) = 1$. Note that ν is unique, and such that $\nu(\lambda_i) = \lambda_{\ell-i}$.

Definition 1. A capacity (or fuzzy measure) is a mapping $\gamma : 2^S \to L$ such that $\gamma(\emptyset) = 0$; $\gamma(S) = 1$; and if $A \subseteq B$ then $\gamma(A) \le \gamma(B)$. The conjugate of γ is the capacity γ^c defined as $\gamma^c(A) = \nu(\gamma(A^c))$, $\forall A \subseteq S$, where A^c is the complement of set A.

The value $\gamma(A)$ can be interpreted as the degree of confidence in a proposition represented by the set A of possible states of the world, or, if S is a set of criteria, the degree of importance of the group of criteria A [27]. In this paper we basically use the first interpretation, unless specified otherwise. We here speak of *qualitative capacity* (or q-capacity, for short) to mean that we only rely on an ordinal structure, not an additive underlying structure.

Remark 1. In fact, even if the scale is encoded by means of numbers in $[0, 1]$, we do not assume these figures represent orders of magnitude, so that their addition or subtraction make no sense. Of course, we could construct a q-capacity using a probability measure P on S, and considering $\{P(A) : A \subseteq S\}$ as the totally ordered set L by renaming the numbers using symbols λ_i. It is always possible to do so using any numerical capacity on S. However, since the symbols λ_i only encode a ranking, we then are unable to distinguish between capacities that yield the same ordering of events (in particular the probability measure P can no longer be distinguished from the many non-additive numerical capacities that yield the same ordering of events as P). So *qualitative* here presupposes that the “distance” between two consecutive λ_i’s can be arbitrary. And in our view, a numerical set-function is a special case of a qualitative one with additional structure in its range.

Important special cases of capacity are possibility and necessity measures.

[1] This paper is based on and extends two previous conference papers [11,21].