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# Coherence in the aggregate: A betting method for belief functions on many-valued events

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ABSTRACT

Betting methods, of which de Finetti's Dutch Book is by far the most well-known, are uncertainty modelling devices which accomplish a twofold aim. Whilst providing an (operational) interpretation of the relevant measure of uncertainty, they also provide a formal definition of *coherence*. The main purpose of this paper is to put forward a betting method for *belief functions* on MV-algebras of many-valued events which allows us to isolate the corresponding coherence criterion, which we term *coherence in the aggregate*. Our framework generalises the classical Dutch Book method.



## 1. Introduction and motivation

Betting methods, of which de Finetti's Dutch Book is by far the most well-known, are uncertainty modelling devices which accomplish a twofold aim. Whilst providing an (operational) interpretation of the relevant measure of uncertainty, they also provide the formal setting to tell apart admissible from inadmissible quantifications of uncertainty. To emphasise the logical, rather than decision-theoretic, nature of this latter aspect, the term *coherence* is often used.[1] The main purpose of this paper is to put forward a betting method for *belief functions* (on many-valued events) which allows us to isolate the corresponding coherence criterion, which we term *coherence in the aggregate*. Since our setting builds on (and extends) de Finetti's method, we begin by recalling his own Dutch Book.

Consider two players, Bookmaker (**B**) and Gambler (**G**) and a finite set of *events of interest* $e_1, \ldots, e_k$ that can only be evaluated as either *true* or *false*. De Finetti's method is best described as an interactive, sequential choice problem (or game), in which the selection of an action, for each player, reveals the player's degree of belief in the corresponding outcome. At the first stage of the game, Bookmaker publishes a *book* $\beta$, i.e. a complete assignment of real numbers $\beta_i \in [0, 1]$ to each event $e_i$. The real number $\beta_i$ is also referred to as the "betting odds" for $e_i$. Once the book has been published, Gambler

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[1] De Finetti, who pioneered betting methods of the kind which will be of interest in this paper, used both the notion of "coherence" and that of "admissibility" depending on whether he wanted to emphasise the logical or decision-theoretic aspect of his analysis, respectively. Compare, for instance, Chapter 3 of [5] with [6].

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chooses stakes $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, one for each event $e_i$, and pays to **B** the amount $\sum_{i=1}^{k} \sigma_i \cdot \beta_i$ in Euros.[2] This makes the monies owed by **B** to **G** depend on a classical valuation (or possible world) $V$ which decides all the relevant events. That is to say, upon $V$ deciding the events of interest, Bookmaker must pay to Gambler $\sum_{i=1}^{k} \sigma_i \cdot V(e_i)$ Euros. Therefore, when all events are decided by some $V$, the total balance in $V$ for **B** is given by the expression:

$$\sum_{i=1}^{k} \sigma_i \cdot \beta_i - \sum_{i=1}^{k} \sigma_i \cdot V(e_i) = \sum_{i=1}^{k} \sigma_i \cdot \big(\beta_i - V(e_i)\big). \tag{1}$$

Clearly, if the result of the above expression (1) is positive, Bookmaker made a profit (in Euros) in $V$, whereas if it is negative, she made a loss in $V$. Since it is reasonable to assume that no Bookmaker would ever aim at losing money, de Finetti's criterion of coherence arises naturally from this setting.

**De Finetti's coherence criterion.** If $e_1, \ldots, e_k$ are events and $\beta$ is a book on them, then $\beta$ is *coherent* if and only if *it does not lead* **B** *to a sure loss*, that is to say, to a total balance for **B** which is negative in every possible world $V$.

De Finetti's celebrated *Dutch Book Theorem* states that a book $\beta$ is coherent if and only if $\beta$ coincides with the restriction to $\{e_1, \ldots, e_k\}$ of a finitely additive and normalised function $P$ mapping elements of the free Boolean algebra generated by the $e_i$'s to $[0, 1]$. It is customary to say that $P$ is a probability measure *extending* $\beta$, or that $\beta$ *extends* to a (finitely additive) probability measure $P$.

A central feature of de Finetti's method is that a possible world $V$ decides completely and unambiguously the truth-value of the events of interest, that is to say, events are for de Finetti, modelled by the semantics of the classical propositional calculus.[3] A practical consequence of this assumption is that $V$ provides **B** and **G** with sufficient information about the (Boolean) events $e_i$'s to compute the value of the total balance in (1). However, it is natural to ask whether de Finetti's method can be extended to characterise coherent belief in those cases in which possible worlds *do not* determine completely whether events of interest are true or false.

Along this line, two generalisations have been proposed by Jaffray [19] and Mundici [25], respectively, to extend de Finetti's betting framework in two different ways. Jaffray investigated betting games where the information possessed by the agent at the time of resolving the uncertainty *may not determine completely* whether the events are true or false. Mundici, on the other hand, investigated betting games where the available information determines the truth value of all the events of interest, but considers a more general semantics than de Finetti's by allowing the *events of interest to be evaluated with degrees of truth between* 0 *and* 1.

Indeed, Jaffray's framework builds on the idea that if a given event $e$ (represented by a sentence of the classical propositional calculus) occurs, then every (non-contradictory event) which follows logically from $e$, also occurs. Jaffray's adaptation of de Finetti's betting method, which he terms a game under *partially resolving uncertainty*, mirrors rather closely the game recalled above. First **B** publishes a book $\beta : e_i \mapsto \beta_i$. Second **G** places stakes $\sigma_1, \ldots, \sigma_k$ on $e_1, \ldots, e_k$ at the betting odds written in $\beta$. Finally, **G** pays **B** for each $e_i$ the amount $\sigma_i \cdot \beta_i$ and **B** gains from **G** the amount $\sigma_i \cdot C_e(e_i)$, where $C_e(e_i) = 1$ if $e_i$ follows from $e$ (under classical propositional logic, i.e. if $\models e \to e_i$), and $C_e(e_i) = 0$ otherwise. Therefore, the total balance for **B** is given by

$$\sum_{i=1}^{k} \sigma_i \big(\beta_i - C_e(e_i)\big). \tag{2}$$

Jaffray calls a book $\beta$ *coherent under partially resolved uncertainty* if it does not lead **B** to incur a sure loss, i.e. if it is not the case that, for every fixed non-contradictory event $e$, $\sum_{i=1}^{k} \sigma_i(\beta_i - C_e(e_i)) < 0$. Finally he shows that this notion of coherence characterises Dempster–Shafer *belief functions* [30] (see Section 2.1) essentially in the same way probability measures are characterised by de Finetti's own notion of coherence:

**Theorem 1.1.** *(See [19].) A book $\beta$ under partially resolved uncertainty on events of interest $e_1, \ldots, e_k \in 2^W$ is coherent iff it can be extended to a belief function on the Boolean algebra $2^W$.*[4]

On the other hand, Mundici extends in [25] de Finetti's coherence criterion to formulas of the infinitely-valued Łukasiewicz calculus. In this setting events are represented by formulas which are evaluated by possible worlds into the real unit

[2] One central condition imposed by de Finetti on the game allows Gambler to choose negative stakes, thereby unilaterally imposing a payoff swap to Bookmaker, who is forced to accept it. So if **G** puts a negative stake $-\sigma_i$ on event $e_i$, she is entitled to receive $\sigma_i \cdot \beta_i$ from **B**. This, and the remaining contractual conditions which underpin de Finetti's Dutch Book are fully analysed, in the language and notation of this paper, in [12]. There, we also emphasise the importance of the (implicit, in de Finetti's framework) assumption to the effect that, at the time of betting, **B** and **G** must be unaware of the truth values of the events involved in the game.

[3] We refer again to [12] for a more detailed analysis of this important point.

[4] Jaffray's original setting is slightly, but immaterially, different from our rendering since he takes Gambler's, rather than Bookmaker's point of view for the calculation of the total balance.