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Geometric views on conflicting mass functions: From distances to angles

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ABSTRACT

Recently, several works have focused on the study of conflict among belief functions with a geometric approach, trying to elaborate on the intuition that distant belief functions are more conflicting than neighboring ones. In this article, I discuss the extent to which the mathematical properties of a metric are compliant with what can be expected from a conflict measure. As this highlights some inconsistencies, numerous other geometric or algebraic objects are investigated for their potential to give a mathematical formulation to the measurement of conflict: divergences, pseudo-sine functions, kernels and semi-pseudo-metrics. Finally, only semi-pseudo-metrics (an intermediate object the nature of which relates to both distances and angles) seem to fit. This is practically confirmed by the fact that the outer plausibility conflict is indeed a semi-pseudo-metric.

1. Introduction

The formal definition of conflict is a recurrent research subject in the belief function community: Although everyone agrees on the fact it is a measure of the level of disagreement between a set of sources of information, it is difficult to provide a clear formula to compute it. With this regard, a new recent trend, composed of numerous works, proposed to tackle this issue in a geometric context, by elaborating on the idea that distant belief functions are more conflicting that neighboring ones. In a recent conference paper [3], I positioned against this trend, by establishing that the mathematical properties of a metric were not desirable for conflict measurement. Yet, in absence of more positive conclusions regarding the existence of other geometric objects that suit conflict measurement, the debate was still open. This is why, the present work extends that of [3]: It enlarges initial investigations for a geometric or algebraic object, that can be of help to measure conflict. Concretely, the methodology is rather simple. Each mathematical object is considered through the list of properties that define it. Then, each property is considered individually, so as to check whether it is a desirable property for conflict measurement. If any of the object properties is not desirable, it cannot be considered to build a sound measure of conflict.

The contributions of the article are multiple. First, there are "negative" conclusions, indicating numerous objects that should not be considered, including distances, which are by far the most used. However, there are also several "positive" conclusions: First, there are six open questions, the resolution of which is mandatory to further investigate geometric interpretations of conflict. Second, the idea that conflict is more related to angles than to distances is sketched. Third, it is proven that all the properties of semi-pseudo-metrics are desirable. Fourth, the outer plausibility conflict from [18] is demonstrated to be a semi-pseudo-metric.

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The article is structured as follows: It starts with some definitions regarding belief function theory, algebra and geometry (Section 2), as well as with a state-of-the-art perspective of the trend (Section 3). Then, various geometric interpretations relying on distances are investigated in Section 4: The main results of [3] are recalled (regarding both *distance-based* and *distance-involved conflicts*) and extended, with the study of Bregman divergences. Afterwards, Section 5 focuses on other geometric interpretations that relate to angles instead of distances: Pseudo-sine functions, their generalized counterparts in the kernel framework, and finally semi-pseudo-metrics. Finally, in Section 6, all these investigations are summarized, discussed and put into perspective.

2. Preliminary definitions

First, some recalls regarding belief function theory [41,44] are given. Then, the algebraic formulation of several geometric notions is given.

2.1. Basics of belief function theory

Definition 1. A *mass function* m over Ω is a mapping $m : \mathcal{P}(\Omega) \to [0, 1]$, with $\mathcal{P}(\Omega)$ the power set of Ω, and such that $\sum_{A\in\mathcal{P}(\Omega)} m(A) = 1$ and $m(\emptyset) = 0$. We denote by $\mathcal{M}(\Omega)$ the set of all mass functions over Ω. A *focal element* A of m is a subset of Ω such that $m(A) \neq 0$. The maximum number of focal elements for mass function $m \in \mathcal{M}(\Omega)$ is noted $n := 2^{|\Omega|} - 1$, where $|.|$ refers to the cardinality operator.

Definition 2. From m can be defined several set-functions [41], the main ones being the *belief*, *plausibility* and *commonality* functions, respectively denoted Bel, Pl and Q and such that, for any non-empty $A \subseteq \Omega$: $Bel(A) = \sum_{\emptyset\neq E\subseteq A} m(E)$, $Pl(A) = \sum_{E\cap A\neq\emptyset} m(E)$, $Q(A) = \sum_{E\supseteq A} m(E)$. The restriction of the plausibility functions on Ω is the *contour function*, noted pl.

Remark 1. m, Bel, Pl and Q are in one-to-one correspondence: they can be recovered from one another.

Definition 3. Several specific mass functions are often defined:

- *vacuous* if $m(\Omega) = 1$, i.e. $Pl(A) = 1$ and $Bel(A) = 0$, $\forall A \subset \Omega$;
- *categorical* if $\exists E \subseteq \Omega$ such that $m(E) = 1$;
- *consonant* if for any focal elements A, B, either $A \subseteq B$ or $B \subseteq A$.

2.2. Interpretations of belief functions

Although there exist numerous interpretations of belief functions theory, they cluster into two groups [21], corresponding to a specific view each. The first one is the *imprecise statistics view*, which was originally proposed by Dempster [14,15], and where mass functions represent some imprecise knowledge about the probability distribution to which obeys a random variable [33]. The second view corresponds to Shafer's original view of belief function, defended in *A mathematical theory of evidence* [41], and in Smets' *transferable belief model* (TBM, [44]). In this view, classically referred to as the *singular view*, $m(A)$ is the mass of belief exactly committed to the hypothesis $\{\omega_0 \in A\}$, where ω_0 is the true value of an ill-known variable $\mathcal{W}$; and the function m represents the uncertainty of an agent about a given but ill-known situation in which only ω_0 holds. Nowadays, the singular view prevails and in a conference such as BELIEF 2012 [17], the majority of the articles undertook it, among which a significant proportion refers to the TBM.

A distinct feature of the singular view is that it is based on an *epistemic model* of uncertainty [5], where sets represent opinions which are not precise enough to focus on a singleton prediction for ω_0. In such an epistemic view, a vacuous mass function models total ignorance, a categorical mass function corresponds to a full conviction on the outcome, yet, with a coarsened description of Ω; and a consonant mass function corresponds to an opinion which bears on a single direction, yet it has a confidence which decreases as the precision increases. Let us also note that, following Smets' works [44], most of the singular interpretations of belief function theory incorporate nowadays the *open-world assumption* of the TBM, where the empty set is authorized to be an additional focal element, which is interpreted as the negation of Ω (i.e. its allocated mass supports hypotheses not formulated in the frame).

Oddly enough, while the singular and the imprecise statistics views differ, their associated combination rules are very similar: the *orthogonal sum* in Dempster's and Shafer's models, and the *conjunctive combination* (an unnormalized version of the orthogonal sum) in the TBM-like interpretations assuming the open-world. The orthonormal sum reads:

$$m_{\textcircled{\scriptsize{n}}}(A) = \frac{1}{1-\mathcal{K}} \sum_{\bigcap_i B_i = A} \prod_i m_i(B_i) \tag{1}$$

where