International Journal of Approximate Reasoning 62 (2015) 27–45

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Decision making under uncertainty comprising complete ignorance and probability

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ARTICLE INFO

Article history:
Received 11 September 2014
Received in revised form 18 April 2015
Accepted 4 May 2015
Available online 7 May 2015

Keywords:
Decision making
Ignorance
Probability
Order of variables

ABSTRACT

This paper investigates a model of decision making under uncertainty comprising opposite epistemic states of complete ignorance and probability. In the first part, a new utility theory under complete ignorance is developed that combines Hurwicz–Arrow's theory of decision under ignorance with Anscombe–Aumann's idea of reversibility and monotonicity used to characterize subjective probability. The main result is a representation theorem for preference under ignorance by a particular one-parameter function – the τ-anchor utility function. In the second part, we study decision making under uncertainty comprising an ignorant variable and a probabilistic variable. We show that even if the variables are independent, they are not reversible in Anscombe–Aumann's sense. This insight leads to the development of a new proposal for decision under uncertainty represented by a preference relation that satisfies the weak order and monotonicity assumptions but rejects the reversibility assumption. A distinctive feature of the new proposal is that the certainty equivalent of a mapping from the state space of uncertain variables to the prize space depends on the order in which the variables are revealed. Explicit modeling of the order of variables explains some of the puzzles in multiple-prior model and the models for decision making with Dempster–Shafer belief function.

1. Introduction

Ignorance and probability are opposite states of knowledge. On the one hand, probability, according to the aleatory interpretation, is a result of knowing all that can be reasonably known about a phenomenon so that its outcome can be modeled as a random event, quantitatively indistinguishable from the randomness of coin toss, roulette spin or radioactive decay. Ignorance, on the other hand, is a singular state of knowledge characterized by knowing nothing or having no reliable information about the phenomenon of interest. Under this extreme state of uncertainty, it is impossible to say that an event except tautology is strictly more likely than another event. We hold the view that the two extreme and opposite states of knowledge form the basis on which other epistemic states are "spanned". This is the main motivation to investigate the models of decision making under uncertainty comprising both ignorance and probability. AI agents would not be truly intelligent without the capability to make decision in situations of ignorance.

Let's start with a motivating example. An investor is considering at the end of 2014 a one-year investment instrument that matures on 1 January 2016. The return on the investment depends on two uncertain variables. The first source of uncertainty is the prospect of a political settlement in country A (e.g. Afghanistan) and the second source of uncertainty is the prospect of 2015 coffee crop in country B (e.g. Brazil). The 2015 coffee harvest, denoted by C, can be either bumper (b)

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http://dx.doi.org/10.1016/j.ijar.2015.05.001

Table 1
The returns on investment.

S/C	b	n	p
s	−3.0%	2.0%	8.0%
$\sim s$	5.0%	1.5%	−4.0%

or normal (n) or poor (p). On the one hand, from extensive historical data, the probability distribution of Brazilian coffee crop in 2015 is estimated to be $(0.46, 0.2, 0.34)$ where the numbers are the chances of having bumper, normal and poor crop respectively. On the other hand, the political settlement variable, denoted by S, is modeled with two possible values: peaceful settlement among fighting factions in 2015 (s) or lack thereof ($\sim s$). The experts whose advice the investor seeks on the political settlement question, offer contradictory opinions and cannot come to any agreement. This underlies the fact that nobody knows the true driving forces behind a political settlement in that region of the world. The returns on the investment are given in the following table.

For example, if there will be a bumper coffee crop and a political settlement then the investment has negative return of −3%. The question for the investor is whether she should invest in the financial instrument or keep her money in the bank that pays interest of 1.2%.

A partial solution for this example is given in Section 2.4 and a complete solution is given in Section 5.1.

This example illustrates how ignorance and probability can coexist in our understanding of real world situations. Ignorance can arise when data and information are scarce, unreliable and contradictory. A player in competitive games may find herself in the state of ignorance if she suspects the information she has about the opponent is intentionally misleading. In the risk assessment practice, ignorance refers to the situations where there is a high level of uncertainty on both the likelihood of events and the consequences of such events [3]. A key condition that differentiates ignorance from uncertainty is the absence of knowledge about the factors that influence the issues [15]. Another condition often associated with the term “ignorance” is the sample space ignorance (SSI) [25] when the decision maker has difficulty in determining the set of alternative states.

Historically, with the development of decision theory under risk in the late 1940s, economists such as Shackle [28], Hurwicz and Arrow [2] started pondering the question how an individual makes decisions if she cannot associate any probability distribution to possible consequences. Earlier, in 1920s, Knight [20] and Keynes [17] came to conclusion that probability theory cannot capture all relevant aspects of uncertainty. In particular, the principle of insufficient reason that produces a probability distribution under ignorance by attributing equal probabilities to alternatives is not always appropriate because the same real situation can be modeled in different ways, with different sets of alternatives. Recently, Gilboa et al. argue that rationality requires a compromise between internal coherence and justification, similarly to compromises that appear in moral dilemmas [12]. Under that view, “it is more rational to admit that one does not have sufficient information to generate a prior than to pretend that one does”.

This paper is structured as follows. Before tackling the problem of decision making under uncertainty involving both ignorance and probability we consider two special cases when uncertainty is either pure probability or complete ignorance. For the former we adopt the standard expected utility. For the latter, in Section 2, a new utility theory under ignorance is developed by pulling together Hurwicz–Arrow's decision theory with Anscombe–Aumann's idea of reversibility of independent variables. The main result is a representation theorem for preference under ignorance. In Section 3 we consider the decision problem that involves both ignorance and probability. We obtain a negative result showing that the reversibility between ignorance and probability is not possible even when they are independent. In Section 4, a solution is proposed that calls for explicit modeling of the order of variables. Section 5 has examples and application. Section 6 contains the discussion of related literature. All proofs are relegated to Appendix A.

Before going into technical presentation, we briefly describe the notation convention used in this paper. The basic objects are *uncertain variables*, denoted by upper case letters such as I, R. A variable has a *domain* denoted by Ω indexed by variable's name. Each variable is accompanied by a measure of uncertainty that gives the variable its type. In this paper, two types are allowed: ignorant variable and probabilistic variable. $\mathcal{O}$ is the set of quantifiable prizes/rewards. $\mathcal{O}$ is assumed to be the real unit interval $\mathbb{R}_{0,1}$.

A mapping from the Cartesian product of the domains of variables to the set of prizes is an act. Acts are denoted by lower case letters such as d, f, g. The behavior of an individual decision maker is described by a preference relation $\succeq$ on acts. We use different notations for acts to signal the type of associated uncertainty. Under Hurwicz–Arrow's assumptions for preference under ignorance, the state space on which an act is defined is not important, the act can be identified with its set of prizes. So, HA acts are denoted by *set* notation $\{x_1, x_2, \ldots\}$. The set of HA acts is denoted by $\mathcal{F}(\mathcal{O})$. For acts that are defined (sequentially) on two ignorant variables we use the notation of set of sets $\{\{x_1, x_2\}, \{y_1, y_2\}, \ldots\}$. The set of such acts is denoted by $\mathcal{F}(\mathcal{F}(\mathcal{O}))$. An act defined on a probabilistic variable is denoted by a *list* of pairs $(p_1{:}x_1, p_2{:}x_2, \ldots, p_n{:}x_n)$ where p_i is the probability of getting reward x_i. Finally, acts that are defined on an ignorant variable and a probabilistic variable are denoted by a *set of lists* (in case the ignorant variable precedes the probabilistic one) or a *list of sets* (in case the order is reversed). In this notation, the investment instrument in Table 1 can be coded as follows:

$$\{(0.46{:}x_{11},\ 0.2{:}x_{12},\ 0.34{:}x_{13}),\ (0.46{:}x_{21},\ 0.2{:}x_{22},\ 0.34{:}x_{23})\}$$