International Journal of Approximate Reasoning 62 (2015) 1–26

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Existence, uniqueness, calculus and properties of triangular approximations of fuzzy numbers under a general condition

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ARTICLE INFO

Article history:
Received 27 June 2014
Received in revised form 24 January 2015
Accepted 7 May 2015
Available online 14 May 2015

Keywords:
Fuzzy number
Triangular fuzzy number
Triangular approximation
Extended approximation

ABSTRACT

We determine the set $\mathcal{P}_t$ of real parameters associated with a fuzzy number which belong to a general family including all the important parameters, with the property that for any given fuzzy number there exists a triangular fuzzy number with the same value of the parameter. We propose a method to compute the nearest triangular fuzzy number which preserves $p \in \mathcal{P}_t$ and we study the properties of identity, scale and translation invariance, additivity and continuity of the obtained approximation operator. We give examples related with recent results in this topic and an application to the nearest triangular approximation of a fuzzy number preserving the value.

1. Introduction

The problem of finding the nearest triangular approximation of a fuzzy number with respect to average Euclidean distance or weighted average Euclidean distance and preserving a parameter was treated in some recent papers (see [2,9,11,29,37–41]). The existence, uniqueness of approximations and properties of additivity, scale and translation invariance, continuity, etc. were studied too. On the other hand, in the last years many methods were elaborated for triangular fuzzy numbers as input data (see, e.g., [12,26–28,30,31,33,35]). Nevertheless, the measured fuzzy numbers in a real life problem or the fuzzy numbers obtained after some preliminary steps are not always triangular fuzzy numbers, such that a method of approximation by triangular fuzzy numbers, preserving a certain characteristic eventually, may be useful.

In the present paper we consider triangular approximations under preservation of a real parameter in the general form $p(A) = al_e(A) + bu_e(A) + cx_e(A) + dy_e(A)$, for any fuzzy number A, where $[l_e(A), u_e(A), x_e(A), y_e(A)]$ is the extended trapezoidal approximation of A and $a, b, c, d \in \mathbb{R}$. It is easy to see that the most important characteristics of fuzzy numbers (expected value, ambiguity, value, width, right and left-hand ambiguity, etc.) and their linear combinations are of this form, therefore many results in [2,9,11,37,38] are immediate consequences. In addition, any new study initiated about a triangular approximation which preserves a parameter, in the general form described above, becomes useless, all the important aspects – existence, uniqueness, calculus, properties – are dealt within the present paper.

We determine the set $\mathcal{P}_t$ of $(a, b, c, d) \in \mathbb{R}^4$ such that for every fuzzy number A there exists a triangular fuzzy number X, with the property $p(A) = p(X)$. We prove that for any fuzzy number A and any $p \in \mathcal{P}_t$, there exists a unique triangular fuzzy number $t_p(A)$ which is the nearest to A with respect to average Euclidean distance between fuzzy numbers such that $p(A) = p\left(t_p(A)\right)$. In addition, we prove that the obtained approximation operators are continuous, translation invariant, nonadditive and satisfy the property of identity, for every $p \in \mathcal{P}_t$. Some recent results in the literature become immediate

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http://dx.doi.org/10.1016/j.ijar.2015.05.004

consequences of our main results in the paper and, on the other hand, the results offer us the possibility to know the properties of triangular approximation operators before effectively calculate them.

The notion of a fuzzy number together with its representations, characteristics and distances, as well as the Karush–Kuhn–Tucker theorem, as an important tool in the approximation of fuzzy numbers, are presented in Section 2. In Section 3 we recall and prove new results related with the extended trapezoidal approximation of a fuzzy number. The existence and uniqueness of the nearest triangular fuzzy number of a fuzzy number under a general condition are studied in Sections 4 and 5. We propose simple methods to compute the triangular approximations in the general case and we exemplify them in Section 6. Section 7 is dedicated to properties of the triangular approximation operators: identity, scale and translation invariance, additivity and continuity. These properties are accepted as essential and, some of them, are difficult to be studied, even in a particular case. We apply the main results of the paper in Section 8, considering the triangular approximation preserving the value.

2. Preliminaries

A fuzzy number A is a fuzzy subset of the real line $\mathbb{R}$ with the membership function A which is (see [18]): normal (i.e. there exists an element x_0 such that $A(x_0) = 1$), fuzzy convex (i.e. $A(\lambda x_1 + (1-\lambda) x_2) \geq \min(A(x_1), A(x_2))$, for every $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$), upper semicontinuous and $cl\{x \in \mathbb{R} : A(x) > 0\}$ is compact, where $cl(M)$ denotes the closure of the set M.

The α-cut, $\alpha \in (0,1]$, of a fuzzy number A is a crisp set defined as

$$A_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}.$$

The support $suppA$ and the 0-cut A_0 of a fuzzy number A are defined as

$$suppA = \{x \in \mathbb{R} : A(x) > 0\}$$

and

$$A_0 = cl\{x \in \mathbb{R} : A(x) > 0\}.$$

Every α-cut, $\alpha \in [0,1]$, of a fuzzy number is a closed interval

$$A_\alpha = [A_L(\alpha), A_U(\alpha)],$$

where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\}$$
$$A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\},$$

for any $\alpha \in (0,1]$ and $[A_L(0), A_U(0)] = A_0$. We denote by $F(\mathbb{R})$ the set of all fuzzy numbers.

Some important characteristics of a fuzzy number (expected interval EI, expected value EV, ambiguity Amb, value Val, width w, left-hand ambiguity Amb_L, right-hand ambiguity Amb_R) of a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0,1]$ were introduced in [13,16,19,24,25] by

$$EV(A) = \frac{1}{2}(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha) \tag{1}$$

$$Amb(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha \tag{2}$$

$$Val(A) = \int_0^1 \alpha(A_U(\alpha) + A_L(\alpha))d\alpha \tag{3}$$

$$w(A) = \int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha \tag{4}$$

$$Amb_L(A) = \int_0^1 \alpha(EV(A) - A_L(\alpha))d\alpha \tag{5}$$