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Rough sets determined by tolerances

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ABSTRACT

We show that for any tolerance R on U, the ordered sets of lower and upper rough approximations determined by R form ortholattices. These ortholattices are completely distributive, thus forming atomistic Boolean lattices, if and only if R is induced by an irredundant covering of U, and in such a case, the atoms of these Boolean lattices are described. We prove that the ordered set RS of rough sets determined by a tolerance R on U is a complete lattice if and only if it is a complete subdirect product of the complete lattices of lower and upper rough approximations. We show that R is a tolerance induced by an irredundant covering of U if and only if RS is an algebraic completely distributive lattice, and in such a situation a quasi-Nelson algebra can be defined on RS. We present necessary and sufficient conditions which guarantee that for a tolerance R on U, the ordered set RS_X is a lattice for all $X \subseteq U$, where R_X denotes the restriction of R to the set X and RS_X is the corresponding set of rough sets. We introduce the disjoint representation and the formal concept representation of rough sets, and show that they are Dedekind–MacNeille completions of RS.

1. Introduction

Rough sets were introduced in [25] by Z. Pawlak. The key idea is that our knowledge about the properties of the objects of a given universe of discourse U may be inadequate or incomplete in the sense that the objects of the universe U can be observed only within the accuracy of indiscernibility relations. According to Pawlak's original definition, an indiscernibility relation E on U is an equivalence relation interpreted so that two elements of U are E-related if they cannot be distinguished by their properties known by us. Thus, indiscernibility relations allow us to partition a set of objects into classes of indistinguishable objects. For any subset $X \subseteq U$, the *lower approximation* $X^{\blacktriangledown}$ of X consists of elements such that their E-class is included in X, and the *upper approximation* $X^{\blacktriangle}$ of X is the set of the elements whose E-class intersects with X. This means that $X^{\blacktriangledown}$ can be viewed as the set of elements certainly belonging to X, because all elements E-related to them are also in X. Similarly, $X^{\blacktriangle}$ may be interpreted as the set of elements that possibly are in X, because in X there is at least one element indiscernible to them. The *rough set* of X is the pair $(X^{\blacktriangledown}, X^{\blacktriangle})$ and the set of all rough sets is

$$RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}.$$

The set RS may be canonically ordered by the coordinatewise order:

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leqslant (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

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In [27] it was proved that *RS* is a lattice which forms also a Stone algebra. Later this result was improved in [6] by showing that *RS* is in fact a regular double Stone algebra. Therefore, *RS* determines also a three-valued Łukasiewicz algebra and a semi-simple Nelson algebra, because it is well known that these three types of algebras can be transformed to each other [23].

In the literature can be found numerous generalizations of rough sets such that equivalences are replaced by relations of different types. For instance, it is known that in the case of quasiorders (reflexive and transitive binary relations), a Nelson algebra such that the underlying rough set lattice is an algebraic lattice can be defined on *RS* [17,18]. If rough sets are determined by relations that are symmetric and transitive, then the structure of *RS* is analogous to the case of equivalences [14]. For a more general approach in the case of partial equivalences, see [22]. There exist also studies in which approximation operators are defined in terms of an arbitrary binary relation – this idea was first proposed in [33]. In [9], expansions of bounded distributive lattices equipped with a Galois connection are represented in terms of rough approximation operators defined by arbitrary binary relations. One may also observe that in the current literature new approximation operators based on different viewpoints are constantly being proposed (see e.g. [1,21] for some recent studies).

In this paper, we assume that indiscernibility relations are tolerances (reflexive and symmetric binary relations). The term *tolerance relation* was introduced in the context of visual perception theory by E.C. Zeeman [36], motivated by the fact that indistinguishability of "points" in the visual world is limited by the discreteness of retinal receptors. One can argue that tolerances suit better for representing indistinguishability than equivalences, because transitivity is the least obvious property of indiscernibility. Namely, we may have a finite sequence of objects $x_1, x_2, \ldots, x_n$ such that each two consecutive objects x_i and x_{i+1} are indiscernible, but there is a notable difference between x_1 and x_n. It is known [12,13] that in the case of tolerances, *RS* is not necessarily a lattice if the cardinality of U is greater than four. Our main goals in this work are to find conditions under which *RS* forms a lattice, and, in case *RS* is a lattice, to study its properties.

As mentioned, originally rough set approximations were defined in terms of equivalences, being bijectively related to partitions. In this paper, we consider tolerances, which are closely connected to coverings. In the literature can be found several ways to define approximations in terms of coverings (see recent surveys in [28,34]), and in this work we connect our approximation operators to some covering-based approximation operators, also.

The paper is organized as follows: In Section 2, we present the definition of rough approximation operators and present their essential properties. In addition, we give preliminaries of Galois connections, ortholattices, and formal concepts. Section 3 is devoted to the rough set operators defined by tolerance relations. Starting from the well-known fact that for any tolerance on U, the pair $({}^\blacktriangle, {}^\blacktriangledown)$ is a Galois connection on the power set lattice of U and characterize rough set approximation pairs as certain kind of Galois connections (F, G) on a power set. We show that $\wp(U)^\blacktriangledown = \{X^\blacktriangledown \mid X \subseteq U\}$ and $\wp(U)^\blacktriangle = \{X^\blacktriangle \mid X \subseteq U\}$ form ortholattices and prove that these ortholattices are completely distributive if and only if R is induced by an irredundant covering of U. Note that distributive ortholattices are Boolean lattices, and a Boolean lattice is atomistic if and only if it is completely distributive. This means that $\wp(U)^\blacktriangledown$ and $\wp(U)^\blacktriangle$ are atomistic Boolean lattices exactly when R is induced by an irredundant covering of U, and we describe the atoms of these lattices. In Section 4, we study the ordered set of rough sets *RS* and show that it can be up to isomorphism identified with a set of pairs $\{(\mathcal{I}(X), \mathcal{C}(X)) \mid X \subseteq U\}$, where $\mathcal{I}$ and $\mathcal{C}$ are interior and closure operators on the set U satisfying certain conditions. We prove that *RS* is a complete lattice if and only if it is a complete subdirect product of $\wp(U)^\blacktriangledown$ and $\wp(U)^\blacktriangle$. We also show that *RS* is an algebraic completely distributive lattice if and only if R is induced by an irredundant covering of U, and in such a case, on *RS* a quasi-Nelson algebra can be defined. The section ends with necessary and sufficient conditions which guarantee that for a tolerance R on U, the ordered set RS_X is a lattice for all $X \subseteq U$, where R_X denotes the restriction of R to the set X and RS_X is the set of all rough sets determined by R_X. Finally, Section 5 is devoted to the disjoint representation and the formal concept representation of rough sets. In particular, we prove that these representations are Dedekind–MacNeille completions of *RS*.

2. Preliminaries: Rough approximation operators, Galois connections, and formal concepts

First we recall from [15] some notation and basic properties of rough approximation operators defined by arbitrary binary relations. Let R be a binary relation on the set U. For any $X \subseteq U$, we denote

$$R(X) = \{y \in U \mid x \, R \, y \text{ for some } x \in X\}.$$

For the singleton sets, $R(\{x\})$ is written simply as $R(x)$, that is, $R(x) = \{y \in U \mid x \, R \, y\}$. It is clear that $R(X) = \bigcup_{x \in X} R(x)$ for all $X \subseteq U$. The *lower approximation* of a set $X \subseteq U$ is

$$X^\blacktriangledown = \{x \mid R(x) \subseteq X\}$$

and X's *upper approximation* is

$$X^\blacktriangle = \{x \mid R(x) \cap X \neq \emptyset\}.$$

Let $\wp(U)$ denote the *power set* of U. It is a complete Boolean lattice with respect to the set-inclusion order. The map ${}^\blacktriangle$ is a complete join-homomorphism on $\wp(U)$, that is, it preserves all unions: