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# On eigenvalues to the Y-bus matrix

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ABSTRACT

The topology of a power system has a profound impact on its reliability. If a power system faces a contingency, for example a loss of a transmission line or a transformer, this contingency might, in worse case, lead to a blackout. Since the Y-bus matrix contains information about the structure, the line impedances, the loading in each bus and is commonly used in power system calculations it can be used to evaluate the topology of the transmission system. This paper reports on the relation between the eigenvalues to the Y-bus matrix and the underlying graph representing the topology of the transmission system. The paper also proposes four different indices' based on the spectrum to the Y-bus matrix and the corresponding Laplacian matrix to be used to evaluate power system topologies. In addition, this paper will also show how the so called algebraic connectivity and the mean impedance in a graph is related and how the mean impedance can be calculated through the eigenvalues to the Laplacian matrix and the Y-bus matrix. In a numerical example, the indices' on the Nordic32 system is presented.


## 1. Introduction

The topology of a power system has a profound impact on its reliability. If a power system faces a contingency, for example a loss of a transmission line or a transformer, this contingency might in worse case lead to a cascading line tripping and/or a voltage collapse [1] resulting in a large scale blackout for the power system. Larger security margins can be applied to reduce this risk. However, it is desirable to fully utilize the capacity of a power system. Further, when evaluating topologies of a power system, extensive and time consuming heuristic methods are used. Three examples are found in [2–4]. Heuristic search methods, such as brutal force, are time consuming when evaluating large scale power systems since the number of states a power system can take is of order $2^n$ where $n$ is the number of failure prone components. An approach to only evaluate the most probable states of a transmission system is made in [2] and an example on a heuristic search method to find critical components in a power system is made in [4]. Here, a component that will cause a blackout or substantially increase the risk for a blackout when removed is defined as a critical component. However, when introducing e.g. a new transmission line in a power system the components that are critical in the new power system will be different from those who are critical before. Since this new transmission line can be connected in a large number of ways, finding how to connect this line in order to minimize the number of critical components, or to maximize the reliability of the power system is an ever more challenging and time consuming task. The complexity of the problem rises quickly if two or three new components are introduced. In order to find good candidates it is therefore necessary to evaluate different topologies efficiently. Here, an efficient method refers to a method with a computational complexity that can be solved in a polynomial time. This is one example of why it is important to efficiently evaluate a topology in a power system.

Graph theory has emerged as a powerful tool to commonly be used in power system calculation of various kinds. Some examples from various authors are found in [5–13].

However, none of these works have investigated the spectrum of the Y-bus matrix which is commonly used in power system calculations. The Y-bus matrix contains information about the topology, impedance and the loading in the system. The Y-bus matrix is obviously not a Laplacian matrix which is well explored in graph theory calculations. However it is only the diagonal elements, i.e. the loading in the system, which differ. For a Laplacian matrix, the algebraic connectivity is a graph theoretical measure for how well connected a graph is. The algebraic connectivity is the second smallest eigenvalue to the Laplacian matrix. The algebraic connectivity of a graph and the corresponding eigenvector have been used in several areas in mathematical research e.g. [14–16]. This particular eigenvalue has been found to give reasonable bounds on several properties on graphs which are very hard to compute, for example the mean distance [17], the diameter [17,18] and the isoperimetric number [19]. These works have showed that the second smallest eigenvalue imposes non-trivial bounds which can be viewed as measures of connectivity for a graph. Therefore, it seems reasonable to investigate the eigenvalues the Y-bus matrix.

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This paper investigates the spectrum to the Y-bus matrix. The paper reports on the relation between the eigenvalues to the Y-bus matrix and the underlying graph representing the topology of the transmission system. The paper also proposes four different indices' based on the spectrum to the Y-bus matrix and the corresponding Laplacian matrix to be used to evaluate power system topologies. In addition, this paper will also show how the algebraic connectivity and the mean impedance in a graph is related and how the mean impedance can be calculated through the eigenvalues to the Laplacian matrix and the Y-bus matrix (i.e. the admittance matrix).

The paper is structured into seven sections. Section 2 gives an introduction to the Laplacian graph theory and shows how the mean impedance is can be calculated via the Moore–Penrose inverse. Sections 3 and 4 shows how the relation between the Y-bus matrix and the Laplacian matrix. Section 5 proposes four indices' divided into two classes based on the calculations in Sections 3 and 4. Section 6 shows this indices' on the Nordic32 bus transmission system. Section 7 gives conclusion.

## 2. The Laplacian matrix

This section will show the relation between the mean impedance and the spectrum to the Laplacian matrix. The standard terminology of graph theory, as it is introduced in most text books, is used. Let the topology of a power system be represented by a graph, $G = (V, E)$, in a natural way, where substations are represented by a set of vertices $V$ and transmission lines by a set of edges $E$ with arbitrary directions. Further, let $C = [c_{vw}]$ be the $|E| \times |V|$ oriented incidence matrix of $G$ with entities

$$c_{vw} = \begin{cases} 1 & \text{if } v \text{ is the terminal vertex of } w, \\ 0 & \text{if } v \text{ and } w \text{ are not incident}, \\ -1 & \text{if } v \text{ is the initial vertex of } w. \end{cases} \quad (1)$$

Moreover, let $D(G)$ be a diagonal matrix where $D = [d_{kk}]$, is a weight on edge $k$ which connects vertices $v$ and $w$. The generalized lapacian matrix for the graph $G$ is then given by

$$Q(G) = C^T DC, \quad (2)$$

where $Q(G) \in \mathbb{C}^{m\times m}$. If the weights in the generalized Laplacian are admittances, the matrix is commonly referred to as the admittance matrix or the Kirchhoff's current matrix. It is well known that these matrices are symmetric, singular and positive semidefinite [20,21]. The eigenvalues and the eigenvectors are referred to as the Laplacian eigenvectors and the Laplacian eigenvalues and is given by

$$Q(G)U_i = \mu_i U_i, \quad (3)$$

where $U_i = (u_{1i}, \ldots, u_{ni})^T$ is the Laplacian eigenvector $i$ and $\mu_i$ is the corresponding eigenvalue $i$. Since

$$Q(G)e = 0, \quad (4)$$

where $e$ is the unit vector i.e. $e = [1, \ldots, 1]^T$, one eigenvalue will always be zero. From the Perron–Frobenius theorem [14], it follows that two eigenvalues are zero, if and only if the graph is not connected. Here, graphs which are connected is considered, that is only one eigenvalue are zero. Let $U$ be the matrix containing the eigenvector, then

$$U^T Q(G)U = \mathrm{diag}(\mu_1, \ldots, \mu_n). \quad (5)$$

Here, diag denotes a diagonal matrix. The element $Q(G)_{ij}$ is given by

$$Q(G)_{ij} = \sum_{k=1}^{n} \mu_k u_{ki} u_{kj}. \quad (6)$$

The matrix $U$ is orthogonal and therefore

$$U^T U = UU^T = I, \quad (7)$$

where $I$ is the unit matrix. If the weights on the edges are admittances, the impedance between any two vertices in an electrical network can be computed via the Moore–Penrose generalized inverse $Q^\dagger$ of the Laplacian matrix. Recall that the Laplacian matrix is singular, and therefore it has no usual inverse. However, the Moore–Penrose generalized inverse is defined and unique for all matrices whose entries are real or complex numbers [22]. The impedance between two nodes $v$ and $w$ in the network is given by

$$\Omega_{vw} = Q(G)^\dagger_{vv} + Q(G)^\dagger_{ww} - 2Q(G)^\dagger_{vw}, \quad (8)$$

here, $\Omega_{vw}$ is the impedance measured between $v$ and $w$. The matrix containing the impedance between all pairs of nodes in an electrical network is referred to as the resistance matrix, $\bar{R} = [r_{vw}]$, and is given by [20]

$$\bar{R} = \mathbf{1}\mathrm{diag}(Q^\dagger) + \mathrm{diag}(Q^\dagger)\mathbf{1}^T - 2Q^\dagger, \quad (9)$$

where $\mathbf{1}$ denotes a column vector with only ones and $\mathrm{diag}(Q^\dagger)$ denotes a row vector consisting of the diagonal entries of $Q^\dagger$. Using Eqs. (6) and (8) the impedance between two vertices can be expressed in terms of eigenvalues and eigenvectors according to

$$\Omega_{vw} = \sum_{k=2}^{n} \frac{1}{\mu_k}(u_{kv} - u_{kw})^2. \quad (10)$$

The mean impedance between all buses can therefore be expressed as

$$\bar{\Omega} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\mu_k}. \quad (11)$$

Here it can be noted that the algebraic connectivity has the largest contribution to the mean impedance in the sum. The mean impedance can be seen as the expected impedance when the impedance is measured between two nodes/buses picked by random.

## 3. The reduced admittance matrix

For an electrical system, the relation between injected current and node potentials is given by

$$Qu = J. \quad (12)$$

Here, $u \in \mathbb{C}$ is a vector containing the voltage magnitude at each bus, $J \in \mathbb{C}$ is a vector for the currents injected into each bus and $Q$ is the admittance matrix, or in other words, the weighted Laplacian matrix to the system. In order for the Kirchhoff's current law to hold, the sum of all elements in $J$ must be zero. There is an infinity amount of solutions to this equation all differ by a constant vector. However, the difference in node potential is always well defined. When considering a power system, one node is usually set as a grounding node and assigned a node potential of 0. When this is done, a reduced admittance matrix can be constructed by removing the corresponding row and column from $Q$. This particular admittance matrix, normally denoted $Y$ in power system calculations, is non-singular and its inverse is therefore well defined. The corresponding potential vector and current vector is obtained by removing the grounding node from the vectors. Let these reduced vectors be denoted $U$ and $I$. The vector $I$ should not be confused with the index matrix. In this formulation, there are no restrictions on $I$ since currents are entering the grounded node. Using this notation, the familiar relation between voltage and injected currents are given by

$$YU = I. \quad (13)$$