Available online at www.sciencedirect.com
ScienceDirect
International Journal of Approximate Reasoning
46 (2007) 511–524
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
www.elsevier.com/locate/ijar

E_T-lipschitzian and E_T-kernel aggregation operators ☆

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Received 22 August 2006; received in revised form 15 January 2007; accepted 15 January 2007
Available online 30 January 2007

Abstract

Lipschitzian and kernel aggregation operators with respect to natural T-indistinguishability operators E_T and their powers are studied. A t-norm T is proved to be E_T-lipschitzian, and is interpreted as a fuzzy point and a fuzzy map as well. Given an archimedean t-norm T with additive generator t, the quasi-arithmetic mean generated by t is proved to be the most stable aggregation operator with respect to T.

Keywords: Aggregation operator; Indistinguishability operator; Natural indistinguishability operator; Lipschitzian condition; Kernel condition

1. Introduction

Lipschitzian conditions are fulfilled by many maps and operators in fuzzy reasoning. They give stability to the system since the similarity or distance between the outputs is bounded by the corresponding one between the inputs. The lipschitzian condition appears for example in the study of fuzzy maps [9], vague algebras [6], fuzzy modifiers and fuzzy logic in the narrow sense [16], fuzzy topology [8], extensionality [9] among others and therefore it deserves a deep study.

☆ Research partially supported by DGICYT project numbers TIC2003-04564 and TIN2006-14311.

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0888-613X/$ - see front matter
doi:10.1016/j.ijar.2007.01.003

Lipschitzian aggregation operators have been studied in [3,4,13,14] considering the usual metric on the unit interval. This paper studies the lipschitzian condition of aggregation operators in a broader sense, i.e. with respect to natural indistinguishability operators E_T and their powers E_T^p (see definitions below) so that an aggregation operator h is E_T^p-lipschitzian when for all $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n \in [0,1] T(E_T^p(x_1, y_1), \ldots, E_T^p(x_n, y_n)) \leqslant E_T(h(x_1, x_2, \ldots, x_n), h(y_1, y_2, \ldots, y_n))$. The meaning is that from similar values we obtain a similar aggregation. The use of E_T and E_T^p assumes the selection of a specific t-norm T and therefore the selection of a particular family of logics where the semantics of the conjunction and the bi-implication are given by T and E_T.

Let us suppose that we are studying or modeling a problem or situation in which a specific t-norm is involved. For instance, a fuzzy system with a t-norm T used to model the logical conjunction (the corresponding disjunction and implication will then probably be related to T). In this case, it seems desirable that the aggregation operators in this particular setting should be related to T in some sense. In particular, it seems more reasonable that lipschitzian conditions of the aggregation operators are defined by means of the bi-implication E_T associated to T than using the usual metric of the unit interval. In fact, as it will be seen in this paper, when T is the Lukasiewicz t-norm, the E_T-lipschitzian condition coincides with the 1-lipschitzian condition with the usual metric on $[0,1]$; not surprisingly, due to the relation between E_T and the usual metric on $[0,1]$ in this case. In other words, the definition of lipschitzian aggregation operator given in [13] tacitly assumes the use of (a logic based on) the Lukasiewicz t-norm.

Among other results, it will be proved that if T is a continuous archimedean t-norm with additive generator t and h_t the quasi-arithmetic mean generated by t $\left(h_t(x_1, x_2, \ldots, x_n) = t^{-1}\left(\frac{t(x_1)+t(x_2)+\cdots+t(x_n)}{n}\right)\right)$, then h_t is the most stable aggregation operator with respect to T (Proposition 3.18). This result not only generalizes the one in [13], but it clarifies its meaning in the sense that it corresponds to the selection of Lukasiewicz t-norm.

The easy to state, but interesting property that the E_T^p-lipschitzian condition is equivalent to the extensionality of the aggregation operator is proven (Proposition 3.10).

Also the t-norm T is not only lipschitzian with respect to E_T, but it can be seen as a fuzzy point and a fuzzy map as well (Propositions 3.20 and 3.22) and an aggregation operator h is greater than or equal to T if and only if h is E_T-lipschitzian.

If in the definition of E_T-lipschitzianity we replace the t-norm T by the minimum $(\mathrm{Min}(E_T^p(x_1, y_1), \ldots, E_T^p(x_n, y_n)) \leqslant E_T(h(x_1, x_2, \ldots, x_n), h(y_1, y_2, \ldots, y_n)))$ we obtain a generalization of the kernel aggregation operators studied in [17,13]. Again, if T is the Lukasiewicz t-norm this definition is equivalent to the one given in the above mentioned references.

After a preliminary section presenting well known definitions and results on t-norms, natural indistinguishability operators and aggregation operators, Section 3 contains the main results of the paper. A last section of concluding remarks closes the paper.

2. Preliminaries

This section contains some results on t-norms and indistinguishability operators that will be needed later on in the paper. Besides well known definitions and theorems, the power T^n of a t-norm is generalized to irrational exponents in Definition 2.4 and given explicitly for continuous archimedean t-norms in Proposition 2.7.