Journal of Symbolic Computation 66 (2015) 84–97

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

An algorithm for computing mixed sums of products of Bernoulli polynomials and Euler polynomials

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ARTICLE INFO

Article history:
Received 22 August 2013
Accepted 22 January 2014
Available online 31 January 2014

Keywords:
Bernoulli polynomials
Euler polynomials
Generating functions
Partial fraction decompositions
Sums of products

ABSTRACT

In this paper, by the methods of partial fraction decomposition and generating function, we give an algorithm for computing mixed sums of products of l Bernoulli polynomials and $k-l$ Euler polynomials, which are of the form

$$T_{n,k}^{\lambda}(y;l,k-l) := \sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geqslant 0}} \prod_{i=1}^{k} \lambda_i^{j_i} \binom{n}{j_1,\ldots,j_k} \times \prod_{p=1}^{l} B_{j_p}(x_p) \prod_{q=l+1}^{k} E_{j_q}(x_q),$$

where $\lambda=(\lambda_1,\ldots,\lambda_k)$, and $\lambda_1,\ldots,\lambda_k$ are nonzero rational numbers. Moreover, some special sums are presented as examples.

1. Introduction and preliminary results

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ play important roles in various branches of mathematics. They are defined by the generating functions

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.1}$$

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0747-7171/$ – see front matter
http://dx.doi.org/10.1016/j.jsc.2014.01.008

They also satisfy the difference equations

$$B_n(x+1)-B_n(x)=nx^{n-1} \quad \text{and} \quad E_n(x+1)+E_n(x)=2x^n \tag{1.2}$$

and multiplication theorems

$$B_n(mx)=m^{n-1}\sum_{k=0}^{m-1} B_n\left(x+\frac{k}{m}\right) \quad \text{for } m=1,2,\ldots, \tag{1.3}$$

$$E_n(mx)=\begin{cases} m^n\sum_{k=0}^{m-1}(-1)^k E_n\left(x+\frac{k}{m}\right) & \text{for } m=1,3,\ldots,\\ -\frac{2}{n+1}m^n\sum_{k=0}^{m-1}(-1)^k B_{n+1}\left(x+\frac{k}{m}\right) & \text{for } m=2,4,\ldots \end{cases} \tag{1.4}$$

(see Abramowitz and Stegun, 1992, Chapter 23, and Comtet, 1974, Section 1.14). Additionally, the rational numbers $B_n=B_n(0)$ and integers $E_n=2^nE_n(1/2)$ are called Bernoulli numbers and Euler numbers, respectively.

Many generalizations of these polynomials have been introduced and studied. For example, the higher order Bernoulli polynomials $B_n^{(\alpha)}(x)$ and higher order Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x and in α, are defined by the generating functions

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{and} \quad \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.5}$$

Clearly, we have $B_n^{(1)}(x)=B_n(x)$ and $E_n^{(1)}(x)=E_n(x)$.

These polynomials and numbers satisfy a large number of identities. In particular, Dilcher (1996) studied the sums of products of arbitrarily many Bernoulli numbers, Bernoulli polynomials, Euler numbers, and Euler polynomials. He found that

$$\begin{aligned} B_n^{(k)}(y)&=\left[\frac{t^n}{n!}\right]\left(\frac{t}{e^t-1}\right)^k e^{yt}=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geqslant 0}}\binom{n}{j_1,\ldots,j_k}B_{j_1}(x_1)\cdots B_{j_k}(x_k)\\ &=(-1)^{k-1}k\binom{n}{k}\sum_{j=0}^{k-1}(-1)^j\left\{\sum_{i=0}^{j}\binom{k-j-1+i}{i}s(k,k-j+i)y^i\right\}\frac{B_{n-j}(y)}{n-j}, \end{aligned} \tag{1.6}$$

where

$$\binom{n}{j_1,\ldots,j_k}=\frac{n!}{j_1!j_2!\cdots j_k!}$$

are the multinomial coefficients, $s(n,k)$ are the Stirling numbers of the first kind, and $y=x_1+\cdots+x_k$. Based on this identity, we have

$$\begin{aligned} B_n^{(2)}(y)&=-(n-1)B_n(y)+n(y-1)B_{n-1}(y),\\ B_n^{(3)}(y)&=\frac{(n-1)(n-2)}{2}B_n(y)-\frac{n(n-2)}{2}(2y-3)B_{n-1}(y)\\ &\quad+\frac{n(n-1)}{2}(y-1)(y-2)B_{n-2}(y). \end{aligned}$$

Similarly, Dilcher's result on sums of products of Euler polynomials is