Journal of Symbolic Computation 75 (2016) 25–55

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

On the computation of the parameterized differential Galois group for a second-order linear differential equation with differential parameters ☆

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ARTICLE INFO

Article history:
Received 8 December 2014
Accepted 10 July 2015
Available online 4 November 2015

MSC:
12H20
34M15
34M03
20H20
13N10
33F10
37K20

Keywords:
Parameterized differential equation
Parameterized Picard–Vessiot theory
Linear differential algebraic group
Unipotent radical
Creative telescoping

ABSTRACT

We present algorithms to compute the differential Galois group G associated via the parameterized Picard–Vessiot theory to a parameterized second-order linear differential equation

$$\frac{\partial^2}{\partial x^2}Y + r_1\frac{\partial}{\partial x}Y + r_0Y = 0,$$

where the coefficients r_1 and r_0 belong to the field of rational functions $F(x)$ over a computable Π-field F of characteristic zero, and the finite set of commuting derivations Π is thought of as consisting of derivations with respect to parameters. This work relies on earlier procedures developed by Dreyfus and by the present author to compute G under the assumption that $r_1 = 0$, which guarantees that G is unimodular. When $r_1 \neq 0$, we reinterpret a classical change-of-variables procedure in Galois-theoretic terms in order to reduce the computation of G to the computation of an associated unimodular differential Galois group H. We establish a parameterized version of the Kolchin–Ostrowski theorem and apply it to give more direct proofs than those found in the literature of the fact that the required computations can be performed effectively. We then extract from these algorithms a complete set of criteria to decide whether any of the solutions to a parameterized second-order linear differential equation is Π-transcendental over the underlying Π-field of $F(x)$. We give

☆ This work was partially supported by a National Science Foundation (NSF) Graduate Research Fellowship (grant 40017-04-06), NSF grant CCF-0952591, and an NSF Alliance Postdoctoral Fellowship.

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http://dx.doi.org/10.1016/j.jsc.2015.11.006
0747-7171/

various examples of computation and some applications to differential transcendence.

1. Introduction

Consider a linear differential equation

$$\delta_x^n Y + \sum_{i=0}^{n-1} r_i \delta_x^i Y = 0 \qquad (1)$$

whose coefficients $r_i \in K = F(x)$ are rational functions in x with coefficients in a computable Π-field F of characteristic zero, δ_x denotes the derivation with respect to x, and $\Pi = \{\partial_1, \dots, \partial_m\}$ is a finite set of pairwise commuting derivations, which we think of as derivations with respect to parameters. Letting $\Delta := \{\delta_x\} \cup \Pi$, we consider K as a Δ-field by setting $\partial_j x = 0$ and $\delta_x \partial_j = \partial_j \delta_x$ for each $1 \le j \le m$.

The parameterized Picard–Vessiot (PPV) theory developed by Cassidy and Singer (2007) associates a differential Galois group G (or PPV group) to (1), in analogy with the classical (or non-parameterized) Picard–Vessiot (PV) theory hinted at by Picard and Vessiot towards the end of the nineteenth century, and put on a firm modern footing by Kolchin (1948) in the middle of the twentieth. The theory of Cassidy and Singer (2007) is a special case of the generalization of Kolchin's strongly normal differential Galois theory (Kolchin, 1953) to the parameterized setting developed by Landesman (2008), and it is also a special case of the differential Galois theory for systems of linear differential-difference equations developed by Hardouin and Singer (2008).

This PPV group G is defined as the group of differential field automorphisms over K of the PPV extension M generated over K by the solutions to (1), together with all their derivatives with respect to Δ. It is shown by Cassidy and Singer (2007) that G admits a structure of linear differential algebraic group (LDAG). These groups, whose study was pioneered by Cassidy (1972), are the differential-algebraic analogues of linear algebraic groups: they are defined as subgroups of $\mathrm{GL}_n(F)$ by a system of Π-algebraic differential equations over F in the matrix entries. The PPV group G in this Galois theory encodes in its differential-algebraic structure the differential-algebraic relations among the solutions to (1).

In retrospect, the classical PV theory corresponds to the special case of the parameterized theory where the set of parameters $\Pi = \emptyset$ is empty. The first general, effective algorithm leading towards the computation of the PV group of (1) for equations of order $n = 2$ is due to Kovacic (1986). Algorithms to compute the PV groups and Liouvillian solutions of second- and third-order linear differential equations are given in Singer and Ulmer, 1993a, 1993b, and an algorithm to compute the PV extension and PV group of a linear differential equation of arbitrary order n with reductive PV group is given in Compoint and Singer (1999). The first complete (though not effective) procedure to compute the PV group of (1) for arbitrary order n was developed by Hrushovski (2002).

Relying on the classification of the differential algebraic subgroups of $\mathrm{SL}_2(F)$ obtained by Sit (1975), and on the algorithm of Kovacic (1986) to compute the Liouvillian solutions of (2) (when they exist), Dreyfus (2014b) developed algorithms to compute the PPV group H associated to

$$\delta_x^2 Y - qY = 0, \qquad (2)$$

where $q \in K$, under the assumption that the field $F = K^{\delta_x}$ is *universal* (Kolchin, 1973, §3.7), which is used by Dreyfus (2014b) in order to interpret F as a field of meromorphic functions on some region. We will instead make the weaker assumption that F is Π-closed (Definition 1), which is required by the PPV theory of Cassidy and Singer (2007).

Although the assumption that F is Π-closed is seldom satisfied in practice, it will play for us only an existential role: if the second-order linear differential equation whose PPV group we wish to compute is defined over $K_0 := F_0(x)$, and F_0 is not Π-closed (e.g., $F_0 = \mathbb{Q}(t_1, \dots, t_m)$, $\Pi = \{\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_m}\}$),