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Dual bases for noncommutative symmetric and quasi-symmetric functions via monoidal factorization

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ABSTRACT

We propose effective constructions of dual bases for the noncommutative symmetric and quasi-symmetric functions. To this end, we use an effective variation of Schützenberger's factorization adapted to the diagonal pairing between a graded space and its dual.

1. Introduction

Originally, "symmetric functions" are thought of as "functions of the roots of some polynomial" (Gelfand et al., 1995). The factorization formula

$$P(X) = \prod_{\alpha \in \mathcal{O}(P)} (X - \alpha) = \sum_{j=0}^{n} X^{n-j} (-1)^j \Lambda_j(\mathcal{O}(P)), \tag{1}$$

where $\mathcal{O}(P)$ is the (multi-)set of roots of P (a polynomial), invites one to consider $\Lambda_j(.)$ as a "multiset (endo)functor"[1] rather than a function $K^n \to K$ (K is a field where P splits). But, here,

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[1] We will not touch here on this categorical aspect.

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$\Lambda_k(X) = 0$ whenever $k > |X|$ and one would like to get the universal formulas *i.e.* which hold true whatever the cardinality of $|X|$. This set of formulas is obtained as soon as the alphabet is infinite and, there, this calculus appears as an art of computing symmetric functions without using any variable. With this point of view, one sees that the algebra of symmetric functions (Macdonald, 1979) comes equipped with many additional structures (comultiplications, λ-ring, transformations of alphabets, internal product, …). As far as we are concerned, the most important of these features is the fact that the (commutative) Hopf algebra of symmetric functions is self-dual. With the exception of self-duality, many features of the (Hopf) algebra of symmetric functions carry over to the noncommutative level (Gelfand et al., 1995). This loss of self-duality has however a benefit: allowing to separate the two sides in the factorization of the diagonal series,[2] thus giving a meaning to what could be considered a complete system of local coordinates for the Hausdorff group of the quasi-shuffle Hopf algebra. Indeed, the elements of the Hausdorff group of the (shuffle or quasi-shuffle) algebras are exactly, through the isomorphism $\mathbf{k}\langle\langle Y\rangle\rangle \simeq (\mathbf{k}\langle Y\rangle)^*$, the characters of the algebra (Bui et al., 2013a; Hoang Ngoc Minh, 2013a, 2013b). Then, letting S be a character and applying $S \otimes \mathrm{Id}$ (necessarily continuous[3]) to the factorization

$$\sum_{w\in Y^*} w \otimes w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(s_l \otimes p_l), \tag{2}$$

S can be decomposed through this complete system of local coordinates:

$$S = \sum_{w\in Y^*} \langle S|w\rangle w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\langle S|s_l\rangle p_l). \tag{3}$$

This fact is better understood when one considers Sweedler's dual of the (shuffle or quasi-shuffle) Hopf algebra $\mathcal{H}$, which contains also, here, the group of characters[4] and its Lie algebra, the space of infinitesimal characters. Such a character is here a series T such that

$$\Delta_*(T) = T \otimes \epsilon + \epsilon \otimes T \tag{4}$$

and one sees from this definition that such a series, as well as the characters, satisfies an identity of the type

$$\Delta_*(S) = \sum_{i=1}^{N} S_i^{(1)} \otimes S_i^{(2)} \tag{5}$$

for some double family $(S_i^{(1)}, S_i^{(2)})_{1\le i\le N}$. Then in (3), the character S is factorized as a product of *elementary* exponentials. This shows firstly, that one can reconstruct a character from its projections onto the free Lie algebra[5] and secondly, that we get a resolution of unity from the process

$$\text{character} \to \text{projection} \to \text{coordinate splitting},$$

$$\text{coordinate splitting} \to \text{exponentials} \to \text{infinite product}.$$

[2] *I.e.* the left hand side of (2) which is an expression of the identity. Note that, in a "diagonal" tensor $w \otimes w$, the left factor serves as a linear form whereas the right factor is a vector. This is why the left hand side will be endowed with the convolution of linear forms and the right hand side with the concatenation.

[3] The series of the form $\sum_{u,v\in Y^*,\mathrm{gr}(u)=\mathrm{gr}(v)} \alpha(u,v) u \otimes v$ where "gr" is a suitable grading (multihomogeneous degree for the shuffle, weight for the quasi-shuffle), form a closed subalgebra of $\mathbf{k}\langle\langle Y^* \otimes Y^*\rangle\rangle$ filtered by the "diagonal" valuation $\mathrm{gr}^{\mathrm{iso}}(u \otimes v) = \mathrm{gr}(u) = \mathrm{gr}(v)$. Then $S \otimes I$, isometric on the monomials, is necessarily continuous.

[4] *I.e.* group-like elements for the dual structure.

[5] For each $l \in \mathcal{L}ynY$ the map $S \mapsto \langle S \mid s_l\rangle\, p_l$ is a projection into the free Lie algebra and these projectors are orthogonal between themselves.