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Quantifier elimination by cylindrical algebraic decomposition based on regular chains ☆

Changbo Chen [a], Marc Moreno Maza [a,b]

[a] *Chongqing Key Laboratory of Automated Reasoning and Cognition, Chongqing Institute of Green and Intelligent Technology, Chinese Academy of Sciences, China*

[b] *ORCCA, Western University, Canada*

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A B S T R A C T

A quantifier elimination algorithm by cylindrical algebraic decomposition based on regular chains is presented. The main idea is to refine a complex cylindrical tree until the signs of polynomials appearing in the tree are sufficient to distinguish the true and false cells. We report an implementation of our algorithm in the `RegularChains` library in Maple and illustrate its effectiveness by examples.

1. Introduction

Quantifier elimination over real closed fields (QE) has been applied successfully to many areas in mathematical sciences and engineering. The following textbooks and journal special issues Hong (1993); Dolzmann et al. (2005); Caviness and Johnson (1998); Basu et al. (2006) demonstrate that QE is one of the major applications of symbolic computation.

It is well known that the worst-case running time for real quantifier elimination is doubly exponential in the number of variables of the input formula, even if there is only one free variable and all polynomials in the quantified input are linear, see J.H. Davenport and C.W. Brown (Brown and Davenport, 2007). It is also well-known that QE based on Cylindrical Algebraic Decomposition (CAD) has a worst-case doubly exponential running time, even when the number of quantifier alternations

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E-mail addresses: changbo.chen@hotmail.com (C. Chen), moreno@csd.uwo.ca (M. Moreno Maza).
URLs: http://www.orcca.on.ca/~cchen (C. Chen), http://www.csd.uwo.ca/~moreno (M. Moreno Maza).

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is constant, meanwhile other QE algorithms are only doubly exponential in the number of quantifier alternations, see J. Renegar (Renegar, 1992) and S. Basu (Basu, 1999).

Despite these theoretical results, the practical efficiency and the range of the applications of CAD-based QE have kept improving regularly since G.E. Collins' landmark paper (Collins, 1975). Today, CAD-based QE is available to scientists and engineers thanks to different software namely QEPCAD,[1] Mathematica,[2] REDLOG,[3] SyNRAC,[4] `RegularChains`.[5]

The work presented here contributes to this effort of making CAD-based QE practically efficient and widely available to the community. The corresponding algorithms were first proposed in our ISSAC 2014 paper (Chen and Moreno Maza, 2014c). The novelty is the use of the theory of regular chains in the context of QE while the implementation in MAPLE can be freely downloaded at the `RegularChains` library website. This work extends our previous results on CAD, which we summarize now.

In Chen et al. (2009), together with B. Xia and L. Yang, we presented a different way of computing CADs, based on triangular decomposition of polynomial systems and therefore on the theory of regular chains. Our scheme relies on the concept of *cylindrical decomposition of the complex space* (CCD), from which a CAD can be easily derived. Since regular chain theory is at the center of this new scheme, we call it RC-CAD. Meanwhile, we shall denote by PL-CAD Collins' projection-lifting scheme for CAD construction.

In Chen and Moreno Maza (2014a), we substantially improved the practical efficiency of the RC-CAD scheme by means of an incremental algorithm for computing CADs; an implementation of this new algorithm, realized within the `RegularChains` library, outperforms PL-CAD-based solvers on many examples taken from the literature.

The purpose of the present paper is to show that RC-CAD, supported by this incremental algorithm, can serve the purpose of real QE. In addition, our implementation of RC-CAD-based QE is competitive with software implementing PL-CAD-based QE.

We turn our attention to the theoretical implication of performing QE by RC-CAD. If extended Tarski formulae are allowed, then deriving QE from a RC-CAD is a straightforward procedure, hence, we shall not discuss it here. In the rest of this paper, for both input and output of QE problems, only polynomial constraints (with rational number coefficients) will be allowed, thus excluding the use of algebraic expressions containing radicals.

In Collins' original work, the augmented projection operator was introduced in order to find a sufficiently large set of polynomials such that their signs alone could distinguish *true* and *false* cells. In Hong (1992), H. Hong produced simple solution formula constructions, assuming that the available polynomials in a CAD were sufficient to generate output formulae.

In his PhD thesis (Brown, 1999), C.W. Brown then introduced ways to add polynomials in an incremental manner and proposed a complete algorithm for producing simple formulae.

It was desirable to adapt Brown's ideas to the context of CADs based on regular chains. However, the many differences between the PL-CAD and RC-CAD schemes were making this adaptation challenging. In the PL-CAD scheme, the key data structure is a set P of projection factors, called the *projection factor set*, meanwhile, in the RC-CAD scheme, it is a tree T encoding the associated CCD (cylindrical decomposition of the complex space). Adding a polynomial f to P corresponds to refining T w.r.t. f (as defined by Algorithm 6 in Chen and Moreno Maza, 2014a). The PL-CAD-property of a *projection-definable* CAD was another key toward practical efficiency in the work of C.W. Brown (Brown, 1999). Its adaptation to the context of RC-CAD, implies that the signs of polynomials in the tree T suffice to solve the targeted QE problem.

After reviewing in Section 2 the basic notions related to RC-CAD (a complete account of which can be found in Chen and Moreno Maza, 2014a) we first adapt in Section 3, the concepts of *projection factor set* and *projection definable*, which were originally introduced by C.W. Brown (Brown, 1999). In

[1] QEPCAD: http://www.usna.edu/CS/~qepcad/B/QEPCAD.html.
[2] Mathematica: http://www.wolfram.com/mathematica/.
[3] REDLOG: http://www.redlog.eu/.
[4] SyNRAC: http://jp.fujitsu.com/group/labs/techinfo/freeware/synrac/.
[5] `RegularChains`: http://www.regularchains.org/.