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Real quantifier elimination for the synthesis of optimal numerical algorithms (Case study: Square root computation) ☆

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ABSTRACT

We report on our on-going efforts to apply real quantifier elimination to the synthesis of optimal numerical algorithms. In particular, we describe a case study on the square root problem: given a real number x and an error bound ε, find a real interval such that it contains $\sqrt{x}$ and its width is less than or equal to ε.

A typical numerical algorithm starts with an initial interval and repeatedly updates it by applying a "refinement map" on it until it becomes narrow enough. Thus the synthesis amounts to finding a refinement map that ensures the correctness and optimality of the resulting algorithm.

This problem can be formulated as a real quantifier elimination. Hence, in principle, the synthesis can be carried out automatically. However, the computational requirement is huge, making the automatic synthesis practically impossible with the current general real quantifier elimination software.

We overcame the difficulty by *(1)* carefully reducing a complicated quantified formula into several simpler ones and *(2)* automatically eliminating the quantifiers from the resulting ones using the state-of-the-art quantifier elimination software.

As the result, we were able to synthesize semi-automatically an optimal quadratically[1] convergent map, which is better than the well known hand-crafted Secant-Newton map. Interestingly, the

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[1] In the ISSAC 2014 paper, we did not require quadratic convergence. In this paper, we require quadratic convergence.

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optimal synthesized map is not contracting as one would naturally expect.

1. Introduction

Real quantifier elimination is a fundamental problem in mathematical logic and computational real algebraic geometry. Furthermore, it naturally arises in many challenging problems in diverse application areas. Due to its importance, there has been extensive research on developing mathematical theories, efficient algorithms, software systems, and applications (just to name a few: Tarski, 1951; Collins, 1975; Arnon, 1988; McCallum, 1988; Grigoriev, 1988; Hong, 1990; Collins and Hong, 1991; Renegar, 1992; Hong, 1992; Liska and Steinberg, 1993; Basu et al., 1996; McCallum, 1999; McCallum, 2001; Anai and Weispfenning, 2001; Brown, 2001a, 2001b; Dolzmann et al., 2004; Strzebonski, 2006; Akbarpour and Paulson, 2010; Strzeboski, 2011; Brown, 2012; Hong and Din, 2012; Bradford et al., 2013; Brown, 2013).

In this paper, we apply real quantifier elimination to the synthesis of optimal numerical algorithms. As a case study, we consider the fundamental operation of computing the square root of a given real number. For the problem, various numerical methods have been developed: Fowler and Robson (1998), Meggitt (1962), Moore (1966), Hart et al. (1968), Cody and Waite (1980), Alefeld and Herzberger (1983), Beebe (1991), Revol (2003), Moore et al. (2009).

We consider an interval version of the problem (Moore, 1966; Alefeld and Herzberger, 1983; Moore et al., 2009): given a real number x and an error bound ε, find an interval such that it contains $\sqrt{x}$ and its width is less than or equal to ε. A typical interval method starts with an initial interval and repeatedly updates it by applying a *refinement map*, say R, on it until it becomes narrow enough (see Algorithm 1). A well known hand-crafted refinement map (called *Secant-Newton*) is given by

$$R^*(I,x) \;\mapsto\; \left[L+\frac{x-L^2}{L+U},\,U+\frac{x-U^2}{2U}\right]$$

Algorithm 1 Interval method for square root.

in: $x>0,\;\; \varepsilon>0$
out: I, an interval such that $\sqrt{x}\in I$ and $\mathrm{width}(I)\le\varepsilon$
  $I\leftarrow[\min(1,x),\max(1,x)]$
  while $\mathrm{width}(I)>\varepsilon$ **do**
    $I\leftarrow R(I,x)$
  return I

which can be easily derived from Fig. 1. A question naturally arises. *Is there any refinement map which is better than Secant-Newton?* In order to answer the question rigorously, one first needs to fix a search space, that is, a family of maps in which we search for a better map. We observe that the Secant-Newton map "scales properly", that is, if we multiply $\sqrt{x}$, L and U by a number, say s, then L' and U' are also multiplied by s. This is due to the fact that the numerators are quadratic forms in $\sqrt{x}$, L and U and the denominators are linear forms. This observation suggests the following choice of a search space: the family of all the maps with the form

$$R_{p,q}:\;[L,U],x\mapsto[L',U']$$

$$L'=L+\frac{x+p_0L^2+p_1LU+p_2U^2}{p_3L+p_4U}\qquad U'=U+\frac{x+q_0U^2+q_1UL+q_2L^2}{q_3U+q_4L}$$