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Factoring linear partial differential operators in n variables

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ABSTRACT

In this paper, we present a new algorithm and an experimental implementation for factoring elements in the polynomial nth Weyl algebra, the polynomial nth shift algebra, and $\mathbb{Z}^n$-graded polynomials in the nth $\underline{q}$-Weyl algebra.

The most unexpected result is that this noncommutative problem of factoring partial differential operators can be approached effectively by reducing it to the problem of solving systems of polynomial equations over a commutative ring. In the case where a given polynomial is $\mathbb{Z}^n$-graded, we can reduce the problem completely to factoring an element in a commutative multivariate polynomial ring.

The implementation in SINGULAR is effective on a broad range of polynomials and increases the ability of computer algebra systems to address this important problem. We compare the performance and output of our algorithm with other implementations in major computer algebra systems on nontrivial examples.

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1. Introduction

In this paper we present a new method and an implementation for factoring elements in the nth polynomial Weyl algebra A_n and the nth polynomial shift algebra. An adaption of these ideas can also be applied to polynomials in the nth $\underline{q}$-Weyl algebra, which is also outlined here. There are numerous important applications for this method, notably since one can view those rings as algebras of operators. For example, given an element $L \in A_n$ and viewing L as a differential operator, one can derive properties of its solution spaces. Especially concerning the problem of finding the solution to the differential equation associated with L, the preconditioning step of factoring L may be helpful.

The new technique heavily uses the nontrivial $\mathbb{Z}^n$-grading on A_n and, to the best of our knowledge, has no analogues in the literature on factorizations for $n \geq 2$. However, for $n = 1$ it is the same grading that lies behind the Kashiwara–Malgrange V-filtration (Kashiwara, 1983; Malgrange, 1983), which is a very important tool in the D-module theory. van Hoeij (1997b) also made use of this technique to factorize elements in the first Weyl algebra with power series coefficients. Notably, for $n \geq 2$, the $\mathbb{Z}^n$-grading we propose is very different from the mentioned $\mathbb{Z}$-grading. Amongst other work, a recent result from Andres (2013) states that the Gel'fand–Kirillov dimension (Gel'fand and Kirillov, 1966) of the $\underline{0}$th graded part of A_n with respect to this $\mathbb{Z}$-grading is in fact $2n - 1$. The Gel'fand–Kirillov dimension of the whole ring A_n is, for comparison, $2n$. The $\underline{0}$th graded part of A_n with respect to the $\mathbb{Z}^n$-grading we propose has Gel'fand–Kirillov dimension n.

Definition 1.1. Let A be a polynomial algebra over a field $\mathbb{K}$ and $f \in A \setminus \mathbb{K}$ be a polynomial. For a fixed totally ordered monomial $\mathbb{K}$-basis of A, the leading coefficient $\mathrm{lc}(f)$ of f is uniquely defined. A **nontrivial factorization** of f is a tuple $(c, f_1, \ldots, f_m)$, where $c \in \mathbb{K} \setminus \{0\}$, $f_1, \ldots, f_m \in A \setminus \{1\}$ are **monic** (i.e. they satisfy $\mathrm{lc}(f_i) = 1$) and $f = c \cdot f_1 \cdots f_m$.

In general, we identify two problems in noncommutative factorization for a given polynomial f: (i) finding one factorization of f, and (ii) finding all possible factorizations of f. Item (ii) is interesting since factorizations in noncommutative rings are not unique in the classical sense (i.e., up to multiplication by a unit), and regarding the problem of solving the associated differential equation one factorization might be more useful than another. We show how to approach both problems here.

A number of papers and implementations have been published on the topic of factorization in algebras of operators over the past few decades. Most of them concentrated on linear differential operators with rational coefficients. Tsarev (1994, 1996) studies the form, number and properties of the factors of a differential operator, extending (Loewy, 1903) and (Loewy, 1906). For differential operators with rational coefficients in more than one variable, Cassidy and Singer (2011) have formulated relations between different factorizations of one operator in terms of differential modules. A general approach to noncommutative algebras and their properties, including factorization, is also presented in the book of Bueso et al. (2003). The authors provide several algorithms and introduce various points of view when dealing with noncommutative polynomial algebras.

In his dissertation, van Hoeij (1996) develops an algorithm to factorize a univariate differential operator. Several papers following his dissertation extend these techniques (van Hoeij, 1997a, 1997b; van Hoeij and Yuan, 2010), and this algorithm is implemented in the `DETools` package of Maple (Monagan et al., 2008) as the standard algorithm for factorization of these operators.

In the REDUCE-based computer algebra system ALLTYPES (Schwarz, 2009), Grigoriev and Schwarz have implemented the algorithm for factoring differential operators they introduced in Grigoriev and Schwarz (2004), which extends the authors' earlier works (Schwarz, 1989) and (Grigor'ev, 1990). As far as we know, this implementation is solely accessible as a web service. Beals and Kartashova (2005) consider the problem of finding a first-order left factor of an element from the second Weyl algebra over a computable differential field, where they are able to deduce parametric factors. Similarly, Shemyakova (2007, 2009, 2010) studies factorization properties of linear partial differential operators.

For special classes of polynomials in algebras of operators, Foupouagnigni et al. (2004) show interesting results about factorizations of fourth-order differential equations satisfied by certain Laguerre–Hahn orthogonal polynomials.