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The prompter method: A treatment for hard-to-solve iterative functional equations

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A R T I C L E I N F O

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A B S T R A C T

The author developed a new method for obtaining formal series solutions to polynomial-like iterative functional equations of the form $\sum_{n=1}^{N} a_n f^n(x) = g(x)$, where $a_n \in \mathbb{R}$, $n = 1, 2, \ldots, N$, f^n is the n-th iterate of an unknown function f and where $g(x)$ is a promptered exponential series, namely, the sum of a Dirichlet series and a linear term called prompter. In this method, a formal composition $f_1 \circ f_2$ of two promptered exponential series f_1 and f_2, where the coefficient of the prompter of f_2 is positive, plays a crucial rôle. We also solve the equation above where $g(x)$ is a promptered trigonometric series.

1. Introduction

Functional equations containing composition of unknown functions have been studied from very long ago, and they are still mysterious in that no general theory like existence or uniqueness theorems is established. The difficulty is explained as follows. If we assume the solution of a functional equation to be a Taylor series or Dirichlet series, and we make functional composition with itself or other transcendental functions, then we often encounter an "unsolvable system of algebraic or transcendental equations" because the comparison of constant term or other term gives an equation containing infinitely many unknown coefficients.

In this paper, for a wide variety of functional equations consisting of composition of exponential and trigonometric functions we present an algorithm called *the prompter method* to obtain formal series solutions, and show its validity by making their numerical evaluations. The key idea is the

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utilization of linear term λx in the solution called *prompter*. If a functional equation satisfies some conditions, then we can put a prompter in the formal series which is supposed to satisfy the equation, and the prompter slides forward the terms when we expand the iteration of the formal series. Then in the comparison of the lowest order we get an algebraic equation consisting only of one unknown, so we can solve it, and in the comparison of the second lowest order, we get an equation consisting of two unknowns, but one of them is already solved, so we can know the second one. Repeating this, we can determine the terms of the solution recursively from lower orders. In the last section, we mention how to deal with functional equations not allowing prompters.

2. Iterative roots

Let I be an interval, N a natural number. For a given function $g: I \to I$, a solution $f: I \to I$ of the iterative functional equation

$$f^N(x) = g(x),$$

where $f^N(x)$ means the N times iterative composition of $f(x)$, is called an N-th iterative root of $g(x)$.

It is difficult to obtain analytic iterative roots of analytic functions. In some cases, iterative roots are obtained by Taylor series expansion of solutions at the fixed point of $g(x)$ (Kuczma et al., 1990; Cheng and Li, 2008), but in general, it is difficult to get (real) analytic solutions. In the case of $g(x) = e^x$, real analytic iterative roots are obtained by Kneser (1950), Belitskii and Tkachenko (2003), and iterative roots not necessarily analytic but having good properties are obtained by Szekeres (1962), Walker (1991) and others. All these results are obtained by solving the Abel equation

$$\Phi(e^x) = \Phi(x) + 1$$

and the N-th iterative root is given by

$$f^{1/N}(x) = \Phi^{-1}\left(\Phi(x) + \frac{1}{N}\right).$$

3. Promptered exponential series

We consider formal series of the form

$$f(x) = \lambda x + \mu + \sum_{n=1}^{M} a_n e^{-p_n x}, \tag{3.1}$$

where $\lambda,\ \mu,\ a_n,\ p_n$ are real numbers, $M = 0,\ 1,\ 2, \ldots, +\infty$ and $0 < p_1 < p_2 < \cdots (\to \infty$ if $M = +\infty)$. Let us call the right-hand side of (3.1) a *promptered exponential series*. It is a Dirichlet series plus an affine term $\lambda x + \mu$, which we call the *affine part of* $f(x)$. We call λx the *prompter* of $f(x)$, μ the *constant term* of $f(x)$. If $\lambda > 0$, then we call λx *positive prompter*, and if $\lambda < 0$, *negative prompter*. If $\lambda = 0$, we call the series *prompter-free*. If $\lambda = \mu = 0$, we call it *affine-free*. The p_n's are called *exponents*. If $M > 0$ and $a_1 \neq 0$, we call p_1 the *initial exponent* of $f(x)$. If $M = 0$, we set the initial exponent to be $+\infty$. Two promptered exponential series are equal if their affine terms coincide and for each real number p, the coefficients of e^{-px} in each series coincide (note that, the coefficient of e^{-px} on the right-hand side of (3.1) is 0 if p does not belong to the set $\{p_1,\ p_2, \ldots\}$). We define the sum of two promptered exponential series and scalar mutiplications of a promptered exponential series in an obvious way. The symbol $\mathbb{S}$ denotes the set of all promptered exponential series. We have the following results.

Lemma 3.1.

(1) *$\mathbb{S}$ is a real vector space.*

(2) *Let $f_i(x),\ i = 0,\ 1,\ 2, \ldots,$ be a sequence in $\mathbb{S}$ and suppose that $f_i(x)$ is affine-free for all $i \geq 1$ and that the initial exponent $p^{(i)}$ of $f_i(x)$ tends to $+\infty$ as $i \to \infty$. Then for each real number $p > 0$, the number of*