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Sparse multivariate function recovery with a small number of evaluations ☆

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ABSTRACT

In Kaltofen and Yang (2014) we give an algorithm based algebraic error-correcting decoding for multivariate sparse rational function interpolation from evaluations that can be numerically inaccurate and where several evaluations can have severe errors ("outliers"). Our 2014 algorithm can interpolate a sparse multivariate rational function from evaluations where the error rate $1/q$ is quite high, say $q = 5$.
For the algorithm with exact arithmetic and exact values at non-erroneous points, one avoids quadratic oversampling by using random evaluation points. Here we give the full probabilistic analysis for this fact, thus providing the missing proof to Theorem 2.1 in Section 2 of our ISSAC 2014 paper. Our argumentation already applies to our original 2007 sparse rational function interpolation algorithm (Kaltofen et al., 2007), where we have experimentally observed that for T unknown non-zero coefficients in a sparse candidate ansatz one only needs $T + O(1)$ evaluations rather than $O(T^2)$ (cf. Candès and Tao sparse sensing), the latter of which we have proved in 2007. Here we prove that $T + O(1)$ evaluations at random points indeed suffice.

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1. Our vector-of-functions recovery setting

We now present the setting of our theorem on the required number of samples for rational function recovery. For the full background including the references to the extant literature and our error-tolerant multivariate rational function interpolation algorithm, its implementation and observed experimental data we refer to our paper Kaltofen and Yang (2014).

We interpolate *a vector* of multivariate *sparse* rational functions with a common denominator:

$$\left[\frac{f^{\langle 1\rangle}}{g}, \ldots, \frac{f^{\langle s\rangle}}{g}\right] \in \mathsf{K}(x_1, \ldots, x_n)^s, \quad g \neq 0. \tag{1}$$

Note that the fractions $f^{\langle\sigma\rangle}/g$ are not necessarily reduced, and they may even have $\mathrm{GCD}(g, \mathrm{GCD}_\sigma(f^{\langle\sigma\rangle})) \neq 1$, because reduction by GCD can affect the sparsity of the fraction, such as $(x_1^d - x_2^d)/(x_1 - x_2)$. We assume that we have for all σ, $1 \le \sigma \le s$, sets of terms $D_f^{\langle\sigma\rangle} \supseteq \mathrm{supp}(f^{\langle\sigma\rangle})$ that constitute maximal sparse supports, and a maximal sparse support set $D_g \supseteq \mathrm{supp}(g)$ for the terms in the common denominator g. See Appendix A for a definition of the support and the meaning of all used symbols. Our algorithms (Kaltofen et al., 2007; Kaltofen and Yang, 2013, 2014) follow the variable-by-variable process by Zippel (1979), which yield those sparse support supersets in each iteration. We suppose that we can evaluate the vector (1) ("probe the black box") at values for the variables, $(x_1, \ldots, x_n) \leftarrow (\xi_{1,\ell}, \ldots, \xi_{n,\ell}) \in \mathsf{K}^n$, for all L evaluations $0 \le \ell \le L-1$, where the $\xi_{\mu,\ell}$ are chosen in a certain way, e.g., selected randomly and uniformly from a finite subset $S \subseteq \mathsf{K}$. As in Kaltofen and Yang (2013), the obtained vector $[\beta_\ell^{\langle 1\rangle}, \ldots, \beta_\ell^{\langle s\rangle}] \in (\mathsf{K} \cup \{\infty\})^s$ can be incorrect in one or more components for $k \le E$ evaluations $\ell = \lambda_1, \ldots, \lambda_k$, that is

$$\forall \kappa, 1 \le \kappa \le k\colon \exists \sigma, 1 \le \sigma \le s\colon \frac{f^{\langle\sigma\rangle}}{g}(\xi_{1,\lambda_\kappa}, \ldots \xi_{n,\lambda_\kappa}) \neq \beta_{\lambda_\kappa}^{\langle\sigma\rangle}, \tag{2}$$

$$\forall \ell \notin \{\lambda_1, \ldots, \lambda_k\}\colon \forall \sigma, 1 \le \sigma \le s\colon \frac{f^{\langle\sigma\rangle}}{g}(\xi_{1,\ell}, \ldots \xi_{n,\ell}) = \beta_\ell^{\langle\sigma\rangle}. \tag{3}$$

Here E is predetermined, for instance from the error rate (Kaltofen and Yang, 2014, Remark 1.1), and the locations of the errors are unknown. As in Kaltofen and Yang (2013) we set *all* components of a vector $= \infty$ if $g(\xi_{1,\ell}, \ldots, \xi_{n,\ell}) = 0$, that even for those components with $f^{\langle\sigma\rangle}(\xi_{1,\ell}, \ldots, \xi_{n,\ell}) = 0$. False vectors full of ∞'s can appear at error locations when $g(\xi_{1,\lambda_\kappa}, \ldots, \xi_{n,\lambda_\kappa}) \neq 0$. We can identify vectors that contain both ∞ and a field element as erroneous. Errors are dealt with by interpolating $(f^{\langle\sigma\rangle}\Lambda)/(g\Lambda)$ à la Kaltofen and Pernet (2013), Kaltofen and Yang (2013) where $\Lambda = (x_{n_1} - \xi_{n_1,\lambda_1}) \cdots (x_{n_1} - \xi_{n_1,\lambda_k})$ is an error locator polynomial for a chosen n_1 with $1 \le n_1 \le n$. We have the maximal supports

$$\left.\begin{aligned} D_{f,E;n_1}^{\langle\sigma\rangle} &= \{\tau x_{n_1}^\nu \mid \tau \in D_f^{\langle\sigma\rangle}, 0 \le \nu \le E\} \supseteq \mathrm{supp}(f^{\langle\sigma\rangle}\Lambda), \\ D_{g,E;n_1} &= \{\tau x_{n_1}^\nu \mid \tau \in D_g, 0 \le \nu \le E\} \quad \supseteq \mathrm{supp}(g\Lambda). \end{aligned}\right\} \tag{4}$$

Now we limit the sparse supports of polynomials with unknown coefficients $\Phi^{\langle\sigma\rangle}$ and Ψ to the term sets (4). From (2) and (3) we obtain linear homogeneous equations for the coefficients of $\Phi^{\langle\sigma\rangle}$, Ψ:

$$\left.\begin{aligned} &\Phi^{\langle\sigma\rangle}(\xi_{1,\ell}, \ldots, \xi_{n,\ell}) - \beta_\ell^{\langle\sigma\rangle}\Psi(\xi_{1,\ell}, \ldots, \xi_{n,\ell}) = 0, \\ &\qquad \text{for } 0 \le \ell \le L-1, 1 \le \sigma \le s \text{ with } \beta_\ell^{\langle\sigma\rangle} \neq \infty, \\ &\Psi(\xi_{1,\ell}, \ldots, \xi_{n,\ell}) = 0, \\ &\qquad \text{for } 0 \le \ell \le L-1 \text{ with } \beta_\ell^{\langle 1\rangle} = \cdots = \beta_\ell^{\langle s\rangle} = \infty, \\ &\quad \text{with } \mathrm{supp}(\Phi^{\langle\sigma\rangle}) \subseteq D_{f,E;n_1}^{\langle\sigma\rangle} \text{ for } 1 \le \sigma \le s,\ \mathrm{supp}(\Psi) \subseteq D_{g,E;n_1}. \end{aligned}\right\} \tag{5}$$

Note that $\Phi^{\langle\sigma\rangle} \leftarrow f^{\langle\sigma\rangle}\Lambda$, $\Psi \leftarrow g\Lambda$ solve (5). We call any solution $(\Phi^{\langle 1\rangle}, \ldots, \Phi^{\langle s\rangle}, \Psi)$ of (5) an interpolant. We seek a (minimal) L and $\xi_{\mu,\ell}$ such that *all* solutions of (5) satisfy

$$\forall \sigma, 1 \le \sigma \le s\colon \Phi^{\langle\sigma\rangle} g = f^{\langle\sigma\rangle}\Psi, \quad \text{with } \mathrm{supp}(\Phi^{\langle\sigma\rangle}) \subseteq D_{f,E;n_1}^{\langle\sigma\rangle},\ \mathrm{supp}(\Psi) \subseteq D_{g,E;n_1}. \tag{6}$$