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Equivalence of differential equations of order one

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ABSTRACT

The notion of strict equivalence for order one differential equations of the form $f(y', y, z) = 0$ with coefficients in a finite extension K of $\mathbb{C}(z)$ is introduced. The equation gives rise to a curve X over K and a derivation D on its function field $K(X)$. Procedures are described for testing strict equivalence, strict equivalence to an autonomous equation, computing algebraic solutions and verifying the Painlevé property. These procedures use known algorithms for isomorphisms of curves over an algebraically closed field of characteristic zero, the Risch algorithm and computation of algebraic solutions. The most involved cases concern curves X of genus 0 or 1. This paper complements work of M. Matsuda and of G. Muntingh & M. van der Put.

1. Introduction and summary

Let K denote a finite extension of $\mathbb{C}(z)$ equipped with the $\mathbb{C}$-linear derivation $'$ given by $z' = 1$. Let $f \in K[S, T]$ be absolutely irreducible and assume that $d := \frac{\partial f}{\partial S} \bmod (f)$ is non-zero. To the first order algebraic differential equation $f(y', y) = 0$ over K we associate the differential algebra $R := K[s, t, \frac{1}{d}] := K[S, T]/(f)[\frac{1}{d}]$, its field of fractions $K(s, t)$, and the pair (X, D), where X is the smooth

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projective algebraic curve over K with function field $K(s,t)$ and D is the derivation on $K(s,t)$ defined by $D(z)=1$, $D(t)=s$. The genus of X will also be called the *genus of* f.

By *a solution of* f we mean a K-linear differential homomorphism $\phi : R \to \mathcal{F}$, where $\mathcal{F}$ is a differential extension of K such that the field of constants of $\mathcal{F}$ is $\mathbb{C}$. More concretely, we may take for $\mathcal{F}$ a finite extension of the field of meromorphic functions on some open connected subset of the Riemann surface of K. Equivalently, a possible $\mathcal{F}$ could be the field of the convergent Laurent series $\mathbb{C}(\{v^{1/m}\})$, where v is a local parameter of a point of the Riemann surface of K and m is a positive integer.

Two first order algebraic differential equations f_1, f_2 over K, inducing pairs (X_1, D_1) and (X_2, D_2) are called *strictly equivalent* if there exists a finite extension $L \supset K$ such that $L \times_K (X_1, D_1) \cong L \times_K (X_2, D_2)$. If f_1, f_2 are strictly equivalent, then for any solution of f_1 we obtain finitely many solutions of f_2 and visa versa.

In the sequel we will rely on algorithms for testing and producing explicit answers to the following questions, briefly discussed in Appendix A. We note that such algorithms exist for the questions (Q1)–(Q4). This does not seem to be the case for question (Q5).

Q1. Are two given curves X_1 and X_2 (over $\overline{K}$ or over $\mathbb{C}$) isomorphic?

Q2. Given is a curve X over K. Does there exist a curve X_0 over $\mathbb{C}$ with $\overline{K} \times_K X \cong \overline{K} \times_{\mathbb{C}} X_0$?

Q3. Does the differential equation $u' = a_0 + a_1 u + a_2 u^2$ with all $a_* \in K$ have solutions in $\overline{K}$? A well known special case is deciding whether $u' = a_1 u$ has a solution in $\overline{K}^*$, see Risch (1970) and Baldassarri and Dwork (1979, § 6).

Q4. Find a basis of $H^0(X, L)$ for a given line bundle L on a given curve X.

Q5. Let E be an elliptic curve over $\mathbb{C}$ and let K be a finite extension of $\mathbb{C}(z)$. Find a basis of the Lang–Néron group (relative Mordell–Weil group) $\mathbb{Q} \otimes_{\mathbb{Z}} E(K)/E(\mathbb{C})$.

We will sketch procedures for strict equivalence of equations, for strict equivalence to an autonomous equation and for having many algebraic solutions. The algebraic treatment of the Painlevé property and related problems for first order equations in the work of M. Matsuda and G. Muntingh & M. van der Put is complemented with an algorithmic approach. Finally, the autonomous equations of any genus are classified.

2. The Painlevé property and algebraic solutions

We recall that a differential equation $f(y', y) = 0$ over a finite extension K of $\mathbb{C}(z)$ is said to have *the Painlevé property* (PP) if the only 'moving singularities' for solutions on the Riemann surface of K are poles. The next theorem provides the link between PP and the definition of 'no moving singularities' in Matsuda (1980).

Theorem 2.1. *(See Muntingh and van der Put, 2007, Proposition 4.2.) Let (X, D) denote the pair associated to a first order equation f over K. Then f has PP if and only if for every closed point $x \in X$, the local ring $O_{X,x}$ is invariant under D.*

Remarks 2.2. (1). There exists *an algorithm verifying PP for any* f. Indeed, the differential algebra $R = K[s, t, \frac{1}{d}]$ is the coordinate ring of an open affine subset of X. Since R is invariant under D one has to investigate the property $D(O_{X,x}) \subset O_{X,x}$ for the finitely many closed points $x \in X$ outside this open subset.

(2). Let $L \supset K$ be a finite extension. Then f over K has PP if and only if f over L has PP. □

Theorem 2.3. *(See Matsuda, 1980; Muntingh and van der Put, 2007.) Let (X, D) denote the pair associated to f over K having PP. There are, after replacing K by a finite extension, three cases for $(K(X), D)$ (where $K(X)$ denotes the function field of X), namely*

(i). $(K(u), (a_0 + a_1 u + a_2 u^2)\frac{d}{du})$ with $a_0, a_1, a_2 \in K$, not all zero.

(ii). $(K(x, y), h \cdot y\frac{d}{dx})$ with $y^2 = x^3 + ax + b$, $a, b \in \mathbb{C}$ a non-singular elliptic curve and $h \in K^*$.

(iii). $(K(X_0), D_0)$ where X_0 is a smooth irreducible projective curve over $\mathbb{C}$ and D_0 is zero on its function field $\mathbb{C}(X_0)$ (which is a subfield of $K(X_0)$).