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Discriminants and nonnegative polynomials

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ABSTRACT

For a semialgebraic set K in $\mathbb{R}^n$, let $P_d(K) = \{f \in \mathbb{R}[x]_{\le d} : f(u) \ge 0 \ \forall u \in K\}$ be the cone of polynomials in $x \in \mathbb{R}^n$ of degrees at most d that are nonnegative on K. This paper studies the geometry of its boundary $\partial P_d(K)$. We show that when $K = \mathbb{R}^n$ and d is even, its boundary $\partial P_d(K)$ lies on the irreducible hypersurface defined by the discriminant $\Delta(f)$ of f. We show that when $K = \{x \in \mathbb{R}^n : g_1(x) = \cdots = g_m(x) = 0\}$ is a real algebraic variety, $\partial P_d(K)$ lies on the hypersurface defined by the discriminant $\Delta(f, g_1, \ldots, g_m)$ of $f, g_1, \ldots, g_m$. We show that when K is a general semialgebraic set, $\partial P_d(K)$ lies on a union of hypersurfaces defined by the discriminantal equations. Explicit formulae for the degrees of these hypersurfaces and discriminants are given. We also prove that typically $P_d(K)$ does not have a barrier of type $-\log \varphi(f)$ when $\varphi(f)$ is required to be a polynomial, but such a barrier exists if $\varphi(f)$ is allowed to be semialgebraic. Some illustrating examples are shown.

1. Introduction

Let K be a semialgebraic set in $\mathbb{R}^n$, and $P_d(K)$ be the cone of multivariate polynomials in $x \in \mathbb{R}^n$ that are nonnegative on K and have degrees at most d, that is,

$$P_d(K) = \left\{f \in \mathbb{R}[x]_{\le d} : f(u) \ge 0 \, \forall u \in K\right\}.$$

Very natural questions arise: What is the boundary of $P_d(K)$? What kind of equation does it satisfy? Can we find a nice barrier function for $P_d(K)$? This paper discusses these issues.

A polynomial $f(x)$ in $x \in \mathbb{R}^n$ is said to be nonnegative or positive semidefinite (psd) on K if the evaluation $f(u) \ge 0$ for every $u \in K$. When $K = \mathbb{R}^n$ and d is even, an $f(x) \in P_d(\mathbb{R}^n)$ is

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called a nonnegative polynomial or psd polynomial. When $K = \mathbb{R}^n_+$ is the nonnegative orthant, an $f(x) \in P_d(\mathbb{R}^n_+)$ is called a co-positive polynomial. Typically, it is quite difficult to check the membership of the cone $P_d(K)$. In the case of $K = \mathbb{R}^n$, for any even $d > 2$, it is NP-hard to check the membership of $P_d(\mathbb{R}^n)$ (e.g., it is NP-hard to check nonnegativity of quartic forms Nesterov (2000) or bi-quadratic forms Ling et al. (2009)). In practical applications, people usually do not check the membership of $P_d(K)$ directly, and instead check sufficient conditions like sum of square (SOS) type representations (a polynomial is SOS if it is a finite summation of squares of other polynomials). There is much work on applying SOS type certificates to approximate the cone $P_d(K)$. We refer the reader to Lasserre (2001); Nie et al. (2006); Parrilo (2003); Parrilo and Sturmfels (2003); Putinar (1993); Reznick (2000); Schmüdgen (1991). However, there is relatively little work on studying the cone $P_d(K)$ and its boundary $\partial P_d(K)$ directly. The geometric properties of $\partial P_d(K)$ are very little known.

When $K = \mathbb{R}^n$ and $d = 2$, $P_2(\mathbb{R}^n)$ reduces to the cone of positive semidefinite matrices, because a quadratic polynomial $f(x)$ is nonnegative everywhere if and only if its associated symmetric matrix $A \succeq 0$ (positive semidefinite). The boundary of $P_2(\mathbb{R}^n)$ consists of f whose corresponding A is positive semidefinite and singular, which lies on the irreducible determinantal hypersurface $\det(A) = 0$. Its degree is equal to the length of matrix A. A typical barrier function for $P_2(\mathbb{R}^n)$ is $-\log\det(A)$. Note that $\det(A)$ is a polynomial in the coefficients of $f(x)$. Do we have a similar result for $P_d(K)$ when $K \neq \mathbb{R}^n$ or $d > 2$? Clearly, when $K = \mathbb{R}^n$ and $d > 2$, we need to generalize the definition of determinants for quadratic polynomials to higher degree polynomials. There has been classical work in this area like Gel'fand et al. (1994). The "determinants" for polynomials of degree 3 or higher are called *discriminants*. The discriminant $\Delta(f)$ of a single homogeneous polynomial (also called form) $f(x)$ is defined such that $\Delta(f) = 0$ if and only if $f(x)$ has a nonzero complex critical point. For a general semialgebraic set K, to study $\partial P_d(K)$, we need to define the discriminant $\Delta(f_0, \dots, f_m)$ of several polynomials $f_0, \dots, f_m$. As we will see in this paper, the discriminant plays a fundamental role in studying $P_d(K)$.

Recently, interests have arisen in the new area of convex algebraic geometry. The geometry of convex (also including nonconvex) optimization problems would be studied by using algebraic methods. There is much work in this field, in areas like maximum likelihood estimation Catanese et al. (2006), k-ellipses Nie et al. (2008), semidefinite programming Nie et al. (2010); Ranestad and Graf von Bothmer (2009), matrix cubes Nie and Sturmfels (2009), polynomial optimization Nie and Ranestad (2009), statistical models and matrix completion Sturmfels and Uhler (2010), convex hulls Ranestad and Sturmfels (in press); Sanyal et al. (2011). In this paper, we study the geometry of the cone $P_d(K)$ by using algebraic methods, and find its new properties.

Contributions The cone $P_d(K)$ is a semialgebraic set, and its boundary $\partial P_d(K)$ is a hypersurface defined by a polynomial equation. To study this hypersurface, we need to define the discriminant $\Delta(f_0, \dots, f_m)$, for several forms $f_0, \dots, f_m$, which satisfies $\Delta(f_0, \dots, f_m) = 0$ if and only if $f_0(x) = \cdots = f_m(x) = 0$ has a nonzero singular solution. This will be shown in Section 3. We prove that when $K = \mathbb{R}^n$ and $d > 2$ is even, $\partial P_d(\mathbb{R}^n)$ lies on the irreducible discriminantal hypersurface $\Delta(f) = 0$, which will be shown in Section 4. We show that when $K = \{x \in \mathbb{R}^n : g_1(x) = \cdots = g_m(x) = 0\}$ is a real algebraic variety, $\partial P_d(K)$ lies on the discriminantal hypersurface $\Delta(f, g_1, \dots, g_m) = 0$ in f, which will be shown in Section 5. We show that when K is a general semialgebraic set, $\partial P_d(K)$ lies on a union of several discriminantal hypersurfaces, which will be shown in Section 6. Explicit formulae for the degrees of these hypersurfaces will also be shown. Generally, we show that $P_d(K)$ does not have a barrier of type $-\log\varphi(f)$ when $\varphi(f)$ is required to be a polynomial, but such a barrier exists if $\varphi(f)$ is allowed to be semialgebraic. For the convenience of readers, we include some preliminaries about elementary algebraic geometry, discriminants and resultants. This will be shown in Section 2.

2. Some preliminaries

2.1. Notation

The symbol $\mathbb{N}$ (resp., $\mathbb{R}$) denotes the set of nonnegative integers (resp., real numbers), and $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. For integer $n > 0$, $[n]$ denotes the set $\{1, \dots, n\}$. For $x \in \mathbb{R}^n$,