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Characterization of rational ruled surfaces

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The algebraic ruled surface is a typical modeling surface in computer aided geometric design. In this paper, we present algorithms to determine whether a given implicit or parametric algebraic surface is a rational ruled surface, and in the affirmative case, to compute a standard parametric representation for the surface.

1. Introduction

Parametric and implicit forms are two main representations of geometrical objects. In computer aided geometric design and computer graphics, people prefer the rational parametric form for modeling design (Farin et al., 2002). On the other hand, in algebraic consideration of computer algebra and algebraic geometry; people usually use the algebraic form. Since there are different advantages of parametric and implicit forms, a nature problem is to convert the forms from one to another. Converting from the implicit form to the parametric one is the parametrization problem. On the converse direction, it is the implicitization problem. There were lots of papers focused on the implicitization problem. Some typical methods were proposed using Gröbner bases (Buchberger, 1985; Cox et al., 1992), characteristic sets (Gao and Chou, 1992a; Wu, 1990), resultants (Dixon, 1908; Pérez-Díaz and Sendra, 2008) and mu-bases (Busé et al., 2009; Chen et al., 2001; Dohm, 2009). However, there still lacks of a method having both completeness in theory and high efficiency in computation.

In general, the parametrization problem is more difficult than the implicitization problem. Only some of the algebraic curves and surfaces have rational parametric representations. For the curves, people have proposed different methods such as parametrization based on resolvents (Gao and Chou,

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1992b), by lines or adjoint curves (Sendra et al., 2007) (see Chapter 4) or using canonical divisor (van Hoeij, 1997). For a general surface, an efficient parametrization algorithm has not been given yet. However, to meet the practical demands, people had to design the parametrization algorithms for some commonly used surfaces. Sederberg and Snively (1987) proposed four parametrization methods for cubic algebraic surfaces. One of them was based on finding two skew lines lying on the surface. Sederberg (1990) and Bajaj et al. (1998) expanded this method. In Wang (2002), a method to parametrize a quadric was given using a stereographic projection. Berry and Patterson (2001) unified the implicitization and parametrization of a nonsingular cubic surface with Hilbert–Burch matrices. These methods were designed for some special surfaces. In Schicho (1998), the author provided a general algorithm that solved the parametrization problem. However, his contributions on theoretical analysis are more than those of practicable computations. Therefore, it is still necessary to find the efficient parametrization algorithm for certain commonly used surfaces.

The ruled surface is an important surface widely used in computer aided geometric design and geometric modeling (see Andradas et al., 2011; Busé et al., 2009; Chen, 2003; Chen et al., 2001; Chen et al., 2011; Dohm, 2009; Izumiya and Takeuchi, 2003; Liu et al., 2006; Li et al., 2008; Pérez-Díaz and Sendra, 2008; Shen and Yuan, 2010; Shen et al., 2012). Using the μ-bases method, Chen et al. (2001) gave an implicitization algorithm for the rational ruled surface. The univariate resultant was also used to compute the implicit equations efficiently (Pérez-Díaz and Sendra, 2008; Shen and Yuan, 2010). For a given rational ruled surface, people could find a simplified reparametrization which did not contain any non-generic base point and had a pair of directrices with the lowest possible degree (Chen, 2003). Busé and Dohm studied the ruled surface using μ-bases (Busé et al., 2009; Dohm, 2009) respectively. Li et al. (2008) computed a proper reparametrization of an improper parametric ruled surface. Andradas et al. presented an algorithm to decide whether a proper rational parametrization of a ruled surface could be properly reparametrized over a real field (Andradas et al., 2011). The ruled surfaces had been used for geometric modeling of architectural freeform design in Liu et al. (2006). The collision and intersection of the ruled surfaces were discussed in Chen et al. (2011), Shen et al. (2012). And Izumiya and Takeuchi (2003) studied the cylindrical helices and Bertrand curves on ruled surfaces. In these papers, the ruled surface was given in standard parametric form $\mathcal{Q}(t_1,t_2) = \mathcal{M}(t_1) + t_2\mathcal{N}(t_1) \in \mathbb{K}(t_1,t_2)^3$. It means the rational ruled surface was preassigned in the discussions. But in general modeling design, such as data fitting, the type of approximate surface may be not known. Then a problem is, for a given parametric surface not being standard form of the ruled surface, how to determine whether it is a ruled surface. If the answer is affirmative, the successive problem is then to find a standard parametric form. In this paper, we would like to consider the determination and reparametrization of the parametric ruled surface.

Go back to parametrization, the implicit surfaces are often introduced in the algebraic analysis. And they can also come directly from modeling design since they have more geometrical features and topologies than those of the parametric surfaces (see Farin et al., 2002; Turk and O'brien, 2002). As we know, there was no paper discussing the parametrization of an implicit rational ruled surface. Here, we would like to consider the parametrization problem of the ruled surface. Precisely, for a given algebraic surface, we first determine whether it is a rational ruled surface, and in the affirmative case, we compute a rational parametrization in standard form. Our discussion is benefited from the standard presentation of the rational ruled surface. Since the parameter t_2 is linear, we can construct a birational transformation to simply the given parametrization. By the linearity again, t_2 is always solvable such that we can project the surface to the rational parametric curve. And these two principal techniques help us to give the determination and (re)parametrization algorithms. The main theorems are all proved constructively, and the algorithms are then presented naturally.

The paper is organized as follows. First, some necessary preliminaries are presented in Section 2. In Section 3, we determine whether a given implicit surface is a rational ruled surface, and in the affirmative case, we compute a rational parametrization in standard form for it. In Section 4, we focus on the parametric surface, including determination and reparametrization. Finally, we conclude with Section 5, where we propose topics for further study.