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Robust toric ideals

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ABSTRACT

We call an ideal in a polynomial ring robust if it can be minimally generated by a universal Gröbner basis. In this paper we show that robust toric ideals generated by quadrics are essentially determinantal. We then discuss two possible generalizations to higher degree, providing a tight classification for determinantal ideals, and a counterexample to a natural extension for Lawrence ideals. We close with a discussion of robustness of higher Betti numbers.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. We call an ideal *robust* if it can be minimally generated by a universal Gröbner basis, that is, a collection of polynomials which form a Gröbner basis with respect to all possible monomial term orders. Robustness is a very strong condition. For instance, if I is robust then the number of minimal generators of each initial ideal is the same:

$$\mu(I) = \mu(\mathrm{in}_< I) \quad \text{for all term orders } < . \tag{1.1}$$

In general, we can only expect an inequality ($\leq$).

For trivial reasons, all monomial and principal ideals are robust. Simple considerations show that robustness is preserved upon taking coordinate projections and joins (see Section 2). However, nontrivial examples of robust ideals are rare. A difficult result of Bernstein and Zelevinsky (1993), Sturmfels and Zelevinsky (1993) (recently extended by Boocher, 2012 and Conca et al., 2013) shows that the ideal of maximal minors of a generic matrix of indeterminates is robust.

In the toric case, questions concerning Gröbner bases have been addressed by many different authors. A classification of the universal Gröbner basis for toric ideals arising from graphs was given

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by Tatakis and Thoma, 2011. For toric varieties of minimal degree, the universal Gröbner basis is encoded by colored partition identities (Petrović, 2008; Bogart et al., 2012). Interest in such problems is aided by the fact that toric ideals enjoy a rich interplay with phylogenetic models, Markov bases, and algebraic statistics. See e.g. Eriksson (2004). Surveying the literature, it is clear that robust toric ideals are rare. The largest known class of robust toric ideals is the set of Lawrence ideals and as far as we are aware, a systematic study of robustness for toric ideals has not yet been undertaken.

Our main result is the following:

Theorem 1.2. *Let F be a set of irreducible binomials that minimally generate an ideal. (Assume that F cannot be partitioned into disjoint sets of polynomials in distinct variables.) If F consists of homogeneous polynomials of degree 2 then the following are equivalent:*

- *The ideal generated by F is robust.*
- *$|F| = 1$ or F consists of the 2×2 minors of a generic $2 \times n$ matrix*

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

  up to a rescaling of the variables.

Furthermore, in higher degree, it is not true that the robustness of (F) implies that it is determinantal (or even Lawrence).

We remark that if the polynomials are not required to be irreducible or homogeneous, the theory of universal Gröbner bases becomes more complicated. Since there is no analogue of Nakayama's lemma, the notion of a minimal generating set is more subtle. In this setting, for example, Dickenstein and Tobis (2012) give an example of a universal Gröbner basis generated by degree two binomials arising from a poset.

Our own motivation for studying robust toric ideals arises primarily from an interest in Gröbner bases of small size. Apart from the seminal result for maximal minors (Bernstein and Zelevinsky, 1993; Sturmfels and Zelevinsky, 1993), there are many cases where one particular Gröbner basis is a minimal generating set. Conca et al. (2006) studied certain classical ideals and determined when they are minimally generated by some Gröbner basis. This is interesting for algebraists, for instance, since one of the most fruitful ways to show an algebra is Koszul has been via the use of G-quadratic ideals — those minimally generated by a quadratic Gröbner basis.

We lastly remark that passing from an ideal to an initial ideal is a particular type of flat degeneration. In this phrasing, robustness is almost equivalent to the property that the minimal number of generators is preserved by these degenerations.[1] This interpretation suggests there might be a geometric interpretation of robustness in terms of the Hilbert scheme.

The paper is organized as follows: in Section 2 we prove our main result, Theorem 1.2 characterizing robust toric ideals generated in degree two. The methods are mainly combinatorial. In Sections 3 and 4 we pose two questions concerning extensions of Theorem 1.2 using Lawrence ideals. We provide negative and positive answers respectively. Section 5 closes with a discussion of "robustness of higher Betti numbers," our original motivation for this project.

2. Quadratic robust toric ideals are determinantal

In the sequel, by a *toric ideal* we will always mean a prime ideal generated minimally by homogeneous binomials with nonzero coefficients in k. By the support of a polynomial we mean the set of variables appearing in its terms.

[1] Note that the equality (1.1) is in general, weaker than robustness. For example, consider the ideal $(x - y, y - z)$.