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Complete intersections in simplicial toric varieties ☆

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ABSTRACT

Given a set $\mathcal{A} = \{a_1, \dots, a_n\} \subset \mathbb{N}^m$ of nonzero vectors defining a simplicial toric ideal $I_{\mathcal{A}} \subset k[x_1, \dots, x_n]$, where k is an arbitrary field, we provide an algorithm for checking whether $I_{\mathcal{A}}$ is a complete intersection. This algorithm does not require the explicit computation of a minimal set of generators of $I_{\mathcal{A}}$. The algorithm is based on the application of some new results concerning toric ideals to the simplicial case. For homogeneous simplicial toric ideals, we provide a simpler version of this algorithm. Moreover, when k is an algebraically closed field, we list all ideal-theoretic complete intersection simplicial projective toric varieties that are either smooth or have one singular point.

1. Introduction

Let k be an arbitrary field and $k[\mathbf{x}] = k[x_1, \dots, x_n]$ and $k[\mathbf{t}] = k[t_1, \dots, t_m]$ two polynomial rings over k. A *binomial* in a polynomial ring is a difference of two monomials. Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a set of nonzero vectors in $\mathbb{N}^m$; each vector $a_i = (a_{i1}, \dots, a_{im})$ corresponds to a monomial $\mathbf{t}^{a_i} = t_1^{a_{i1}} \cdots t_m^{a_{im}}$ in $k[\mathbf{t}]$. The *toric set* Γ determined by $\mathcal{A}$ is the subset of the affine space $\mathbb{A}_k^n$ given parametrically by $x_i = u_1^{a_{i1}} \cdots u_m^{a_{im}}$ for all $i \in \{1, \dots, n\}$, i.e.,

$$\Gamma = \left\{ \left(u_1^{a_{11}} \cdots u_m^{a_{1m}}, \dots, u_1^{a_{n1}} \cdots u_m^{a_{nm}}\right) \in \mathbb{A}_k^n \,\middle|\, u_1, \dots, u_m \in k \right\}.$$

The kernel of the homomorphism of k-algebras $\varphi : k[\mathbf{x}] \to k[\mathbf{t}]$; $x_i \mapsto \mathbf{t}^{a_i}$ is called the *toric ideal* of Γ and will be denoted by $I_{\mathcal{A}}$. For every $b = (b_1, \dots, b_n) \in \mathbb{N}^n$ one sets the $\mathcal{A}$-degree of the

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monomial $\mathbf{x}^b \in k[\mathbf{x}]$ as $\deg_{\mathcal{A}}(\mathbf{x}^b) := b_1 a_1 + \cdots + b_n a_n \in \mathbb{N}^m$. One says that a polynomial $f \in k[\mathbf{x}]$ is $\mathcal{A}$-homogeneous if its monomials have the same $\mathcal{A}$-degree. By Sturmfels (1996, Corollary 4.3), it is an *$\mathcal{A}$-homogeneous binomial* ideal, i.e., $I_{\mathcal{A}}$ is generated by $\mathcal{A}$-homogeneous binomials. According to Sturmfels (1996, Lemma 4.2), the height of $I_{\mathcal{A}}$ is equal to $n - \dim(\mathbb{Q}\mathcal{A})$. By Villarreal (2001, Corollary 7.1.12), if k is an infinite field, $I_{\mathcal{A}}$ is the ideal $I(\Gamma)$ of the polynomials vanishing in Γ. The variety $V(I_{\mathcal{A}}) \subset \mathbb{A}_k^n$ is called an *affine toric variety*.

The ideal $I_{\mathcal{A}}$ is a *complete intersection* if $\mu(I_{\mathcal{A}}) = \mathrm{ht}(I_{\mathcal{A}})$, where $\mu(I_{\mathcal{A}})$ denotes the minimal number of generators of $I_{\mathcal{A}}$. Equivalently, $I_{\mathcal{A}}$ is a complete intersection if there exists a set of $s = n - \dim(\mathbb{Q}\mathcal{A})$ $\mathcal{A}$-homogeneous binomials $g_1, \ldots, g_s$ such that $I_{\mathcal{A}} = (g_1, \ldots, g_s)$. The problem of determining complete intersection toric ideals has a long history; see the introduction of Morales and Thoma (2005) and the references there.

We denote by $\mathrm{Cone}(\mathcal{A})$ the cone spanned by $\mathcal{A}$, i.e., $\mathrm{Cone}(\mathcal{A}) = \{\sum_{i=1}^n \alpha_i a_i \mid \alpha_i \in \mathbb{R}_{\geq 0}\}$. An *extreme ray* of $\mathrm{Cone}(\mathcal{A})$ is a set $F_a := \mathrm{Cone}(\{a\})$ such that whenever $x, y \in \mathrm{Cone}(\mathcal{A})$ satisfy that $x + y \in F_a$, then $x, y \in F_a$. It is well known, see for example Cox et al. (2011, Lemma 1.2.15), that a set $\{a'_1, \ldots, a'_s\} \subset \mathbb{R}^m$ is a minimal set of generators of $\mathrm{Cone}(\mathcal{A})$ if and only if the extremal rays of $\mathrm{Cone}(\mathcal{A})$ are $F_{a'_1}, \ldots, F_{a'_s}$. Thus, the number of extremal rays of $\mathrm{Cone}(\mathcal{A})$ is $\geq \dim(\mathbb{Q}\mathcal{A})$. When equality holds the toric ideal $I_{\mathcal{A}}$ is said to be a *simplicial toric ideal* and $V(I_{\mathcal{A}})$ an *affine simplicial toric variety*. If $I_{\mathcal{A}}$ is homogeneous, then $V(I_{\mathcal{A}}) \subset \mathbb{P}_k^{n-1}$ is called a *simplicial projective toric variety*.

The aim of this work is to obtain and implement an efficient algorithm for checking whether a simplicial toric ideal is a complete intersection that does not require the explicit computation of a minimal set of generators of the ideal.

This work follows the line we began in Bermejo et al. (2007a), where we proposed an algorithm for checking whether the toric ideal of an affine monomial curve is a complete intersection. That algorithm was based on the ideas introduced in Bermejo et al. (2005) and we implemented it in Singular (Decker et al., 2012), giving rise to the library `cimonom.lib` (Bermejo et al., 2007b). This work is a non-trivial generalization of Bermejo et al. (2007a) for simplicial toric ideals and gives rise to the Singular library `cisimplicial.lib` (Bermejo and García-Marco, 2012), which generalizes, outperforms and substitutes our previous `cimonom.lib`.

For proving correctness of our algorithm we will use the following direct consequence of Rosales (1997, Theorem 1.4): if $\mathcal{A}$ is a gluing of $\mathcal{A}_1$ and $\mathcal{A}_2$ and both $I_{\mathcal{A}_1}$ and $I_{\mathcal{A}_2}$ are complete intersections, then so is $I_{\mathcal{A}}$. Recall that $\mathcal{A}$ is a gluing of $\mathcal{A}_1$ and $\mathcal{A}_2$ if $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ and there exists $\alpha \in \mathbb{N}\mathcal{A}_1 \cap \mathbb{N}\mathcal{A}_2$ such that $\mathbb{Z}\alpha = \mathbb{Z}\mathcal{A}_1 \cap \mathbb{Z}\mathcal{A}_2$.

It is worth pointing out that Fischer, Morris and Shapiro provided in Fischer et al. (1997) a theoretical characterization of the property of being a complete intersection in toric ideals by proving that whenever $I_{\mathcal{A}}$ is a complete intersection, there exist $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ such that $\mathcal{A}$ is a gluing of $\mathcal{A}_1$ and $\mathcal{A}_2$ and both $I_{\mathcal{A}_1}, I_{\mathcal{A}_2}$ are complete intersections. This result was improved in García-Sánchez and Rosales (1995) for the particular case of simplicial toric ideals. Our approach to the problem of determining complete intersection simplicial toric ideals is different in nature to that of Fischer et al. (1997) and García-Sánchez and Rosales (1995).

The main achievement of this work is Algorithm CI-simplicial, which receives as input any set $\mathcal{A} \subset \mathbb{N}^m$ such that $I_{\mathcal{A}}$ is a simplicial toric ideal and returns True if $I_{\mathcal{A}}$ is a complete intersection or False otherwise. Moreover, whenever $I_{\mathcal{A}}$ is a complete intersection, the algorithm provides without any extra effort a minimal set of $\mathcal{A}$-homogeneous generators of $I_{\mathcal{A}}$. This algorithm is based on the application of some new results concerning complete intersection toric ideals, namely Theorems 2.4 and 3.2, to the simplicial case. Correctness of Algorithm CI-simplicial is proved in Theorem 4.3, which is the main result of this paper.

The structure of the paper is the following.

Sections 2 and 3 are devoted to present two techniques on complete intersection toric ideals. In Section 2 we prove Theorem 2.4. This result is based on the idea of associating to $\mathcal{A}$ another set $\mathcal{A}_{red} \subset \mathbb{N}^m$ that can be either empty or defines a toric ideal $I_{\mathcal{A}_{red}}$ satisfying that $I_{\mathcal{A}}$ is a complete intersection if and only if either $\mathcal{A}_{red} = \emptyset$ or $I_{\mathcal{A}_{red}}$ is a complete intersection. Moreover, in case $\mathcal{A}_{red}$