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The connections between three-way and classical concept lattices

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ABSTRACT

The model of three-way concept lattices, a novel model for widely used three-way decisions, is an extension of classical concept lattices in formal concept analysis. This paper systematically analyses the connections between two types of three-way concept lattices (object-induced and attribute-induced three-way concept lattices) and classical concept lattices. The relationships are discussed from the viewpoints of elements, sets and orders, respectively. Furthermore, the necessary and sufficient conditions used to construct three-way concepts on the basis of classical concepts are proved, the algorithms building three-way concept lattices on the basis of classical concept lattices are presented. The obtained results are finally demonstrated and verified by examples.

1. Introduction

Three-way concept analysis (3WCA), proposed by Qi et al. in 2014 [21], is a combination of three-way decisions (3WD) [32–35] and formal concept analysis (FCA) [4,28]. The three-way concept lattice in 3WCA not only extends the classical (two-way) concept lattice in FCA, but also provides a novel model for three-way decision-making.

As a unified and discipline-independent framework for the decision-making with three options, the concept of three-way decisions was suggested by Yao in 2012 [35]. Since then, more and more studies on the theory, model and application of 3WD have been widely investigated in various areas of scientific research and engineering [3,5,6,10,14–16,21,24,30,31,36–41,43,44]. The essential idea of 3WD is to divide a universe into three pair-wise disjoint regions according to evaluations of a set of given criteria [35]. For the three different regions, different rules may be constructed to make three-way decisions.

Formal concept analysis, initially aiming at an application framework for lattice theory, was proposed by Wille in 1982 [28]. Now it has evolved into an efficient tool for data analysis, and been applied to various fields successfully [1,4,7–9,11–13,17–23,25–27,29,42]. Two key components in FCA include formal concepts and concept lattices. A concept lattice is an ordered hierarchical structure of all the formal concepts, which are constructed from a formal context with an object universe and an attribute universe. A formal concept is determined by a pair of elements, that is, an object subset (extension) and an attribute subset (intension). It expresses the semantics of "*jointly possessed*" between the object subset and the attribute subset in the formal context.

Applying the idea of 3WD to FCA, we obtained three-way concepts and three-way concept lattices [21]. Be similar to a formal concept in FCA, a three-way concept is also constituted of an extension and an intension. However, different from a formal concept, the extension (or the intension) of a three-way concept is equipped with two parts: positive one and negative one. These two parts are used to express the semantics "*jointly possessed*" and "*jointly not possessed*" in a formal context, respectively. On the basis of a three-way concept, one can divide the object (or attribute) universe into three regions to make three-way decisions.

As an extension of classical formal concepts, it is obvious that three-way concepts must be related to formal concepts with respect to a formal context. Furthermore, other interesting problems arise: (1) what are the connections between the set of all three-way concepts and that of all formal concepts? (2) what are the connections between the three-way and classical concept lattices? and (3) on the basis of these connections and classical concept lattices, how to build three-way concept lattices? These issues will be investigated in the paper.

This paper is organised as follows. Section 2 briefly reviews some basic definitions and notions related to FCA and 3WCA. In

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Section 3, the connections between object-induced three-way concept lattices and classical concept lattices are studied firstly, then an acquisition approach to object-induced three-way concept lattices based on classical concept lattices is proposed and the corresponding algorithm is presented. Section 4 investigates the connections between attribute-induced three-way concept lattices and classical concept lattices, and presents an acquisition approach to attribute-induced three-way concept lattices based on classical concept lattices and the corresponding algorithm. Finally, conclusions are drawn in Section 5.

2. Preliminaries

In this paper, we refer to Reference [2] for the notions about lattices and order, especially for the direct product of two ordered sets (two lattices). We denote by $\mathcal{P}(\cdot)$ the power set of a set and by $\mathcal{DP}(\cdot)$ the product $\mathcal{P}(\cdot) \times \mathcal{P}(\cdot)$. The operations, intersection $\cap$ and union $\cup$, on $\mathcal{DP}(\cdot)$ are defined componentwise by using standard set operations.

In the rest of this section, we briefly reviews the basic notions regarding to FCA and 3WCA.

2.1. Basic definitions in FCA

Definition 2.1 ([4]). A formal context is a triple (U, V, R), which consists of two sets U and V and a relation R between U and V. The elements of U are called the objects and the elements of V are called the attributes of the context. For $x \in U$ and $a \in V$, we write $(x, a) \in R$ as xRa, and say that the object x has the attribute a, or alternatively, the attribute a is possessed by the object x.

With respect to a formal context (U, V, R), the following operators can be defined.

Definition 2.2 ([4]). For $X \subseteq U$ and $A \subseteq V$, a pair of operators, $^* : \mathcal{P}(U) \to \mathcal{P}(V)$ and $^* : \mathcal{P}(V) \to \mathcal{P}(U)$, are defined by

$$X^* = \{a \in V \mid \forall x \in X (xRa)\},$$

$$A^* = \{x \in U \mid \forall a \in A (xRa)\}.$$

They have the following properties [4]: if $X,\ X_1,\ X_2 \subseteq U$ are sets of objects and $A,\ A_1,\ A_2 \subseteq V$ are sets of attributes, then

(C1) $X \subseteq X^{**}$ and $A \subseteq A^{**}$,
(C2) $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*$ and $A_1 \subseteq A_2 \Rightarrow A_2^* \subseteq A_1^*$,
(C3) $X^* = X^{***}$ and $A^* = A^{***}$,
(C4) $X \subseteq A^* \Leftrightarrow A \subseteq X^*$,
(C5) $(X_1 \cup X_2)^* = X_1^* \cap X_2^*$ and $(A_1 \cup A_2)^* = A_1^* \cap A_2^*$,
(C6) $(X_1 \cap X_2)^* \supseteq X_1^* \cup X_2^*$ and $(A_1 \cap A_2)^* \supseteq A_1^* \cup A_2^*$.

Given a formal context (U, V, R), a formal concept (for short, a concept) is defined to be a pair (X, A) of an object subset $X \subseteq U$ and an attribute subset $A \subseteq V$ where $X^* = A$ and $A^* = X$. X is called the extension and A is called the intension of the concept (X, A) [4]. It is easy to know that both (X^{**}, X^*) and (A^*, A^{**}) are formal concepts.

The family of all the concepts forms a complete lattice that is called the concept lattice of (U, V, R) and denoted by $CL(U, V, R)$. Meanwhile, we write the set of the extensions and the set of the intensions of all the concepts as $CL_E(U, V, R)$ and $CL_I(U, V, R)$, respectively.

Let $(X_1, A_1),\ (X_2, A_2) \in CL(U, V, R)$ be formal concepts. Then the partial order is defined by [4]:

$$(X_1, A_1) \le (X_2, A_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow A_1 \supseteq A_2.$$

The infimum and supremum are given by [4]

$$(X_1, A_1) \wedge (X_2, A_2) = (X_1 \cap X_2, (A_1 \cup A_2)^{**}),$$

$$(X_1, A_1) \vee (X_2, A_2) = ((X_1 \cup X_2)^{**}, A_1 \cap A_2).$$

The two operators * defined above are called positive operators in Reference [21]. Correspondingly, Reference [21] also defines a pair of negative operators as follows.

Definition 2.3 ([21]). Let (U, V, R) be a formal context. For $X \subseteq U$ and $A \subseteq V$, a pair of negative operators, $^{\bar{*}} : \mathcal{P}(U) \to \mathcal{P}(V)$ and $^{\bar{*}} : \mathcal{P}(V) \to \mathcal{P}(U)$, are defined by

$$X^{\bar{*}} = \{a \in V \mid \forall x \in X (\neg(xRa))\} = \{a \in V \mid \forall x \in X (xR^c a)\},$$

$$A^{\bar{*}} = \{x \in U \mid \forall a \in A (\neg(xRa))\} = \{x \in U \mid \forall a \in A (xR^c a)\}.$$

Here $R^c = (U \times V) - R$.

It is obvious that the negative operators of (U, V, R) are just the positive operators of (U, V, R^c). Hence, negative operators have the similar properties as positive ones. Suppose $X,\ X_1,\ X_2 \subseteq U$ and $A,\ A_1,\ A_2 \subseteq V$, then

(NC1) $X \subseteq X^{\bar{*}\bar{*}}$ and $A \subseteq A^{\bar{*}\bar{*}}$,
(NC2) $X_1 \subseteq X_2 \Rightarrow X_2^{\bar{*}} \subseteq X_1^{\bar{*}}$ and $A_1 \subseteq A_2 \Rightarrow A_2^{\bar{*}} \subseteq A_1^{\bar{*}}$,
(NC3) $X^{\bar{*}} = X^{\bar{*}\bar{*}\bar{*}}$ and $A^{\bar{*}} = A^{\bar{*}\bar{*}\bar{*}}$,
(NC4) $X \subseteq A^{\bar{*}} \Leftrightarrow A \subseteq X^{\bar{*}}$,
(NC5) $(X_1 \cup X_2)^{\bar{*}} = X_1^{\bar{*}} \cap X_2^{\bar{*}}$ and $(A_1 \cup A_2)^{\bar{*}} = A_1^{\bar{*}} \cap A_2^{\bar{*}}$,
(NC6) $(X_1 \cap X_2)^{\bar{*}} \supseteq X_1^{\bar{*}} \cup X_2^{\bar{*}}$ and $(A_1 \cap A_2)^{\bar{*}} \supseteq A_1^{\bar{*}} \cup A_2^{\bar{*}}$.

Given a formal context (U, V, R), a concept induced by its negative operators is called a N-concept. Obviously, $(X^{\bar{*}\bar{*}}, X^{\bar{*}})$ and $(A^{\bar{*}}, A^{\bar{*}\bar{*}})$ are N-concepts. The corresponding concept lattice is then denoted by $NCL(U, V, R)$. The order, infimum and supremum of $NCL(U, V, R)$ are defined in the same way as those of $CL(U, V, R)$. Analogously, $NCL_E(U, V, R)$ and $NCL_I(U, V, R)$ represent the set of extensions and the set of intensions of N-concepts, respectively. It should be noted that both $CL(U, V, R)$ and $NCL(U, V, R)$ are classical concept lattices.

2.2. Basic definitions in 3WCA

Combining positive operators and negative operators together, we can obtain the following two pairs of three-way operators.

Definition 2.4 ([21]). Let (U, V, R) be a formal context. Given $X \subseteq U$ and $A, B \subseteq V$, a pair of object-induced three-way operators, $^{\lessdot} : \mathcal{P}(U) \to \mathcal{DP}(V)$ and $^{\gtrdot} : \mathcal{DP}(V) \to \mathcal{P}(U)$, are defined by

$$X^{\lessdot} = \left(X^*, X^{\bar{*}}\right),$$

$$(A, B)^{\gtrdot} = \left\{x \in U \mid x \in A^* \text{ and } x \in B^{\bar{*}}\right\} = A^* \cap B^{\bar{*}}.$$

We abbreviate them as OE-operators.

Definition 2.5 ([21]). Let (U, V, R) be a formal context. Given $A \subseteq V$ and $X, Y \subseteq U$, a pair of attribute-induced three-way operators, $^{\lessdot} : \mathcal{P}(V) \to \mathcal{DP}(U)$ and $^{\gtrdot} : \mathcal{DP}(U) \to \mathcal{P}(V)$, are defined by

$$A^{\lessdot} = \left(A^*, A^{\bar{*}}\right),$$

$$(X, Y)^{\gtrdot} = \left\{a \in V \mid a \in X^* \text{ and } a \in Y^{\bar{*}}\right\} = X^* \cap Y^{\bar{*}}.$$