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Special types of coverings and axiomatization of rough sets based on partial orders

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ABSTRACT

Covering rough sets are a generalization of Pawlak rough sets, in which a partition of the universal set induced by an equivalence relation is replaced by a covering. In this paper, covering rough sets are transformed into generalized rough sets induced by binary relations. The paper discusses three theoretical topics. First, we consider a special type of covering in which the neighborhoods form a reduction of the covering, and we obtain necessary and sufficient conditions for neighborhoods in a covering form a reduction of the covering. Second, we study another special type of covering, and give conditions for the covering lower and upper approximations to be dual to each other. Finally, we give an axiomatic system that characterizes the lower and upper approximations of rough sets based on a partial order.

1. Introduction

Rough set theory, proposed by Pawlak [14,15] in 1982, is a useful tool for dealing with vagueness and granularity in information systems. Pawlak rough set theory is built on equivalence relations. However, equivalence relations impose restrictions and limitations in many applications [3,5,7,8,13,24,26]. Thus, one of the main directions of research in rough set theory is the generalization of rough approximations. Many extensions have been made to the theory by replacing equivalence relations with notions such as arbitrary binary relations [19,21,26], fuzzy relations [11], and coverings [16,20,23,30,33] of the universal sets. Zakowski [29] first extended Pawlak rough sets by using coverings of the domain rather than partitions.

Many types of covering rough set model have been proposed and investigated since Zakowski [29] extended Pawlak rough sets; see, for example, [4,31,33]. Bonikowski et al. [1] formulated conditions for the existence of operations on covering rough sets. Diker and Ugur [4] introduce relations between coverings and dicoverings within the framework of rough sets. In [17], Restrepo et al. investigated properties of approximation operators, and proved a characterization for pairs of approximation operators that are both dual and adjoint. In [18], they continued their study of covering approximation operators, and organized covering rough sets into 6 dual pairs. More recently, Yun et al. [28] characterized the conditions for a neighborhood $\{N(x)|x \in U\}$ to form a partition of the universal set U. The problem of reduction arises in many practical applications and is an important area of research in database and information systems [9]. A useful method for reduction of information systems is the rough set theory approach that depends on partitioning the universal set of objects using equivalence relations. As we know, reduction is also a significant topic in covering rough sets. The concept of reduction in coverings was introduced by Zhu and Wang [32], who moreover gave a procedure for finding a reduction for a covering. In contrast with the conditions from [28] for a neighborhood $\{N(x)|x \in U\}$ to form a partition of universal set U, a neighborhood $\{N(x)|x \in U\}$ is not in general a reduction of a covering. A natural question to ask is whether we can characterize the conditions under which $\{N(x)|x \in U\}$ forms a reduction of C. This is clearly an interesting mathematical question, and in this paper we answer it affirmatively. In addition, neighborhoods are also an important concept in topology. The answer to this question provides more insight into the topological structure of covering rough sets. We hope that the results in this paper can be used to select useful features and eliminate superfluous attributes in covering information systems.

The motivation of the paper is the question above. In this paper we only consider one type of covering rough sets based on neighborhoods. Our purpose is three theoretical topics. First, we give necessary and sufficient conditions under which $\{N(x)|x \in U\}$ is the reduction of a covering of U. Second, we consider conditions under which $\underline{C}$ and $\overline{C}$ are dual to each other. Finally, we study axiomatic approaches to rough sets, which are important for

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understanding their mathematical structure and may help with developing methods for real applications. There has been much effort made in researching axiomatic approaches [11,22,25]. However, up until now, rough sets based on a partial order have not been axiomatized. Partial orders are an important type of relation. We give an axiomatic system for rough sets induced by a partial order in this paper.

Throughout the paper there is no requirement for the universal set to be finite. That is, we work over a fixed universal set U, where unless otherwise stated the cardinality of U is infinite.

The remainder of this paper is organized as follows. In Section 2, we review standard definitions of rough sets, covering rough sets, and reductions of coverings. In Section 3, we study interesting properties of neighborhoods $\{N(x)|x \in U\}$ and the approximations $\underline{N}$ and $\overline{N}$ induced by a covering C of U. We also obtain conditions for neighborhoods in a covering to form a reduction. In Section 4, we consider conditions under which $\overline{C}$ and $\underline{C}$ are dual to each other. In Section 5, we investigate the properties of posets, and give an axiomatic system for rough sets based on a partial order. Finally, we conclude the paper in Section 6.

2. Preliminaries

This section reviews briefly the fundamental notation and notions based on generalized rough sets, covering rough sets, and neighborhoods. We refer to [11,32,33] for details.

2.1. Generalized rough sets

Although there are many different types of generalized rough sets, we only consider the type proposed by Yao [26], which is the most commonly used one in rough set theory. Let U be a universal set and R be an arbitrary relation on U. The left R- and right R-relative sets for an element x in U are defined as

$$l_R(x) = \{y|y \in U, yRx\} \text{ and } r_R(x) = \{y|y \in U, xRy\}.$$

Clearly, for each $x \in U, l_{R^{-1}}(x) = r_R(x)$ and $r_{R^{-1}}(x) = l_R(x)$, where R^{-1} is the inverse of R. Recall that the following terminology: (1) R is reflexive if $x \in r_R(x)$ for each $x \in U$; (2) R is symmetric if $l_R(x) = r_R(x)$ for each $x \in U$; (3) R is antisymmetric if xRy and yRx implies that $x = y$; (4) R is transitive if $x \in r_R(y)$ implies that $r_R(x) \subseteq r_R(y)$; (5) R is an equivalence relation if R is reflexive, symmetric and transitive; and (6) R is a partial order if R is reflexive, antisymmetric and transitive.

Yao and Yao [27] considered three different equivalent forms based on element, granule, and subsystem definitions. This paper only considers the following element-based definition of generalized rough sets.

Definition 2.1 26. Let U be a universal set and R be an arbitrary binary relation on U. For each subset X of U, we define two subsets,

$$\underline{R}(X) = \{x|r_R(x) \subseteq X\} \text{ and}$$
$$\overline{R}(X) = \{x|r_R(x) \cap X \neq \emptyset\},$$

called the lower and upper approximations of X respectively.

We need the following three simple but important properties in this paper.

(1) $\overline{R}(\{x\}) = l_R(x)$ for each $x \in U$.
(2) Distributivity with respect to union: $\overline{R}(\cup_{i\in I}X_i) = \cup_{i\in I}\overline{R}(X_i)$ for any given index set I, where $X_i \subseteq U$.
(3) $\overline{R}(X) = \cup_{x\in X}l_R(x)$ for each $X \subseteq U$.

2.2. Covering rough sets and its topologies

As generalizations of Pawlak rough sets, many types of covering rough set approximations were proposed by [2]. However, we only focus on one generalization in this paper.

Definition 2.2 (1,32). Let U be a given universal set, and $C = \{K|K \subseteq U\}$ be a family of nonempty subsets of U. If $\cup_{K\in C}K = U$, then C is called a covering of U. If C is a covering of U, then we call the ordered pair (U, C) a covering approximation space. $N(x) = \cap\{K|K \in C, x \in K\}$ is called a neighborhood of an element $x \in U$. The minimal description of an element $x \in U$ is defined as $Md(x) = \{K|x \in K \in C \wedge (\forall S \in C, x \in S \subseteq K \Rightarrow K = S)\}$.

Throughout this paper, for any given covering approximation space (U, C), we define the binary relation R on U to satisfy

$$l_R(x) = N(x)$$

for each $x \in U$. We use this relation to build a relationship between generalized rough sets and covering rough sets. It is easily verified that R is a quasiorder (a reflexive and transitive binary relation) on U. It is well known that there is a one-to-one correspondence between quasiorders on U and quasi-discrete topologies [10] on U. Clearly, $\underline{R}$ is an interior and $\overline{R}$ a closure for a quasi-discrete topology $\sigma_R = \{X|X \subseteq U, \underline{R}(X) = X\}$ on U.

Bonikowski et al. [1] and Zhu [32] proposed the following covering approximations.

Definition 2.3. Let (U, C) be a covering approximation space. For any $X \subseteq U$, the lower and upper approximations of X are defined as $\underline{C}(X) = \cup\{K|K \in C, K \subseteq X\}$ and $\overline{C}(X) = \cup\{N(x)|x \in X - \underline{C}(X)\} \cup \underline{C}(X)$, respectively.

Proposition 2.1. *Let (U, C) be a covering approximation space. Then $K = \cup_{x\in K}N(x)$ for all $K \in C$.*

Proof. Suppose that $K \in C$ and $x \in K$. Because $x \in N(x)$, we have $K \subseteq \cup_{x\in K}N(x)$. Conversely, $\cup_{x\in K}N(x) \subseteq K$ is clear. So $K = \cup_{x\in K}N(x)$, and the proposition is proved. □

Reduction is an important concept in covering rough set theory, and many algorithms for reduction in coverings have been proposed [32]. The following definition of reduction in a covering was proposed by Zhu and Wang [32].

Definition 2.4 (32). Let C be a covering of U with $K \in C$.

(1) If K is a union of some elements in $C - \{K\}$, we say that K is reducible in C. Otherwise, K is irreducible.
(2) If every element of C is irreducible, we say that C is irreducible. Otherwise, C is reducible.
(3) If $C' \subseteq C$ and C' is an irreducible covering of U, then C' is called a reduction of C.

Note that each covering C has a unique reduction denoted by $red(C)$. As mentioned above, both $red(C)$ and $\{N(x)|x \in U\}$ are coverings of U. In general, $red(C) \neq \{N(x)|x \in U\}$, that is, $\{N(x)|x \in U\}$ is not the reduction of C shown in the following example.

Example 2.1. Let $U = \{1, 2, 3, 4\}, K_1 = \{1, 2\}, K_2 = \{2, 3\}$, and $K_3 = \{3, 4\}$. Then $C = \{K_1, K_2, K_3\}$ is a covering of U. Moreover, $N(1) = \{1, 2\}, N(2) = \{2\}, N(3) = \{3\}$, and $N(4) = \{3, 4\}$. $N(2) \notin C$, so $\{N(x)|x \in U\}$ is not a subset of C, and therefore, $\{N(x)|x \in U\}$ is not the reduction of C.

Observe that, for any given covering C of $U, red(C)$ is also a covering of U, and thus the lower approximation $\underline{red(C)}$ and the