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An entropy measure definition for finite interval-valued hesitant fuzzy sets

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ABSTRACT

In this work, a definition of entropy is studied in an interval-valued hesitant fuzzy environment, instead of the classical fuzzy logic or the interval-valued one. As the properties of this kind of sets are more complex, the entropy is built by three different functions, where each one represents a different measure: fuzziness, lack of knowledge and hesitance. Using all, an entropy measure for interval-valued hesitant fuzzy sets is obtained, quantifying various types of uncertainty.

From this definition, several results have been developed for each mapping that shapes the entropy measure in order to get such functions with ease, and as a consequence, allowing to obtain this new entropy in a simpler way.

1. Introduction

The fuzzy logic was introduced by Zadeh in 1971 (see [32]), becoming a generalization of the classical set theory which has been widely studied since then up to now. The goal of this approach is to represent certain properties that are not possible to be dealt with by the classical logic. It has been applied to a wide range of topics, such as protection of privacy (see [21,22]) or image processing (see [2,23]).

A fuzzy set is characterized by a membership function that depends on the expert that shapes it. In order to overcome this problem, generalizations of the fuzzy sets were carried out. In particular, the interval-valued fuzzy sets were introduced by Sambuc in 1975 (see [25]), where the membership function gives for each point not a single value but an interval. Atanassov's intuitionistic fuzzy sets, developed by Atanassov in 1986 (see [1]), are another generalization where a set is associated to both a membership and a non-membership function. A greater extension of the fuzzy sets are the 2-type fuzzy sets, given by Zadeh in 1975 (see [33]), where for each point, the membership function is defined over the referential $[0,1]$.

However, type-2 fuzzy sets are difficult to work with, so in 2009 hesitant sets were introduced by Torra (see [27,28]) as an intermediate kind of fuzzy sets. The membership function of a hesitant set assigns a subset of the closed interval $[0,1]$ instead of a fuzzy set to each point. This property makes them more manageable than type-2 fuzzy sets. In fact, these sets were already introduced by Grattan-Guinness [15] in 1976, with the name of set-valued fuzzy sets. However, Torra provided functional definitions of union and intersection for such sets which were not considered by Grattan-Guinness. This type of sets is currently a rising researching topic, due to the possibilities that they provide (see [5,14,30]), and specially, in decision making (see [10,16,29]). Different extensions of this hesitant sets have been developed lately (see [24]). In our paper, the used and studied generalization is the finite interval-valued hesitant fuzzy sets, given by Pérez et al. in 2014 (see [20]).

The study of entropy measures in the fuzzy set theory also became an important part of the research, firstly defined by De Luca and Termini in 1972 (see [11]), whose aim is to quantify the uncertainty associated to a fuzzy set. This concept has been adapted to other types of fuzzy sets, such as Atanassov's intuitionistic fuzzy sets (see [18]), interval-valued fuzzy sets (see [6]) or even interval-valued hesitant fuzzy sets (see [14]).

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Nevertheless, the existing definition of entropy for interval-valued hesitant fuzzy sets in [14] only reflects one type of uncertainty, associated to how distant a set is from a union of crisp sets. Our proposal along the work is to define a new entropy measure for interval-valued hesitant fuzzy sets, where three types of uncertainty are reflected through three mappings, instead of the classical concept of just one function for one type of uncertainty associated. In addition, several results have been developed in order to obtain such mappings with ease, and as a result, the entropy measure can be obtained with simpler conditions. Note that this has also been the approach in [18] for the Atanassov intuitionistic fuzzy setting.

The remainder of the paper is structured as follows: the following section is splitinto three subsections with preliminary concepts about fuzzy sets, hesitant fuzzy sets and entropy and dissimilarity measures respectively. Section 3 details the study related to the new definition of entropy in a interval-valued hesitant environment. In Section 4 the main conclusions of this work are highlighted.

2. Preliminaries

Necessary concepts to understand the definition of entropy proposed in this work are given in this section. It has been split into three subsections. General basic concepts about the fuzzy logic are explained in the former. The used generalization of fuzzy sets, the hesitant fuzzy sets, are developed in the second one. In the latter, the definitions of entropy and dissimilarity measure in different environments are given.

2.1. Fuzzy sets and its extensions

The concepts about the usual types of fuzzy sets can be found in a wide range of sources, such as [9]. These types of sets are important in order to understand the utility provided by the hesitant fuzzy sets, starting with the definition of the classic fuzzy set, which was given for the first time by Zadeh (see [32]).

Definition 1. Let X be a non-empty set. Given the membership function:

$$\mu_A : X \to [0,1],$$

then, the set $A = \{(x, \mu_A(x)) | x \in X\}$ is a fuzzy set in X.

Given $x \in X$, the value $\mu_A(x)$ is called membership degree of x.

Remark 2. $FS(X)$ denotes the set of all fuzzy sets in X.

In addition to the definition of fuzzy set, the following concepts are introduced in order to develop the forthcoming results.

Definition 3. Given $A, B \in FS(X)$, with their membership functions μ_A and μ_B respectively:

- The complement of A with respect to the standard negation, which is denoted by A^c, is the fuzzy set given by $A^c = \{(x, 1 - \mu_A(x)) | x \in X\}$.
- The partial ordering relation used for fuzzy sets is given by:
$$A \leqslant B \iff \mu_A(x) \leqslant \mu_B(x), \quad \forall x \in X.$$
- The set $\xi \in FS(X)$ is called equilibrium set if it is defined as $\xi = \{(x, 0.5) | x \in X\}$.

The interval-valued fuzzy sets are a generalization of the fuzzy sets, where an interval instead of just one value is associated to each point. This kind of sets were developed by Sambuc (see [25]).

Definition 4. Let X be a non-empty set. Given the membership function:

$$\mu_A : X \to L([0,1]),$$

where $L([0,1])$ denotes the family of all closed subintervals of $[0,1]$, then, the set $A = \{(x, \mu_A(x) = [\mu_A(x)^L, \mu_A(x)^U]) | x \in X\}$ is an interval-valued fuzzy set in X.

Remark 5. $IVFS(X)$ denotes the set of all interval-valued fuzzy sets in X.

Some useful concepts are introduced in the following definition.

Definition 6. Given $A, B \in IVFS(X)$, with their membership functions μ_A and μ_B respectively,

- The complement of A with respect to the standard negation, which is denoted by A^c, is given by $A^c = \{(x, \mu_{A^c}(x)) | x \in X\}$, where $\mu_{A^c}(x) = [1 - \mu_A(x)^U, 1 - \mu_A(x)^L], \ \forall x \in X$,
- The partial ordering relation used in our paper for interval-valued fuzzy sets, is well known and can be found in several sources such as [3,19]. It is given by:
$$A \leqslant B \iff \mu_A(x) \leqslant_I \mu_B(x), \quad \forall x \in X,$$
where $\forall x \in X$
$$\mu_A(x) \leqslant_I \mu_B(x) \iff \mu_A(x)^L \leqslant \mu_B(x)^L \text{ and } \mu_A(x)^U \leqslant \mu_B(x)^U,$$
- The set $A = \{(x, [0,1]) | x \in X\}$ is called the pure interval-valued fuzzy set.

The concept of pure interval-valued fuzzy set is obtained directly from the concept of pure Atanassov intuitionistic fuzzy set introduced in [18], taking into account the mathematical duality between both concepts (see [26]).

In addition, type-2 fuzzy sets were also developed by Zadeh. They represent a generalization of the classical notion of fuzzy set (see [33]).

Definition 7. Let X be a non-empty set. Given the membership function:

$$\mu_A : X \to FS([0,1]),$$

then, $A = \{(x, \mu_A(x)) | x \in X\}$ is a type-2 fuzzy set in X.

Remark 8. $T2FS(X)$ denotes the set of all type-2 fuzzy sets in X. As we will work on a subset of $T2FS(X)$, we are not going to comment any operation on type-2 fuzzy sets, in order to avoid unnecessary explanations.

In the next subsection, basic concepts about hesitant fuzzy sets are studied. This type of sets represents an intermediate step between the interval-valued fuzzy sets and the 2-type fuzzy sets, which makes them interesting to study and work with. The reason lies in the fact that the type-2 fuzzy sets are hard to handle and use, while the hesitant fuzzy sets have properties that make them more manageable. Furthermore, all the results obtained in a hesitant environment can be quickly adapted to other types of sets, such as interval-valued fuzzy sets and the classical fuzzy sets, since they are a generalization of them.

2.2. Hesitant fuzzy sets

Hesitant fuzzy logic, recently defined by Torra in [27,28], was first introduced by Grattan-Guinnes in [15], with the name of