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# Neighborhood systems and covering approximation spaces

Yu-Ru Syau [a], En-Bing Lin [b,*]

[a] Department of Information Management, National Formosa University, Yunlin 63201, Taiwan
[b] Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48859, USA

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ABSTRACT

The approximation theory is studied via a generalization of pretopological and topological neighborhood systems, called total pure reflexive neighborhood systems. In the framework of such neighborhood systems, the definition of pre-topological equivalence is formulated in terms of the concept of "neighborhoods". We show that the so-called lower and upper approximations are pre-topological invariants and dual to each other, and we construct the corresponding generalized topological closure via the upper approximation.

We regard a given covering as a special form of a total pure reflexive neighborhood system, called a covering neighborhood system, by assigning to each object $x$ the nonempty family $CN(x)$ of all members of the covering that contain $x$. We then investigate the approximation structure of the covering by introducing two pairs of lower and upper approximations: one pair treats each $CN(x)$ as a local base, the other pair as a local subbase. We obtain optimal lower and upper approximations of a covering of an approximation space.



## 1. Introduction

The lower and upper approximations [17,18] are fundamental concepts of rough set theory. They are defined in an approximation space which is an ordered pair $(U, \beta)$ with a fixed nonempty set $U$ of all the objects under consideration and a partition $\beta$ on $U$.

Zakowski [28] considered a generalization of $(U, \beta)$ by replacing the partition with a covering $\mathcal{C}$ on the set $U$; the ordered pair $(U, \mathcal{C})$ was referred to as a covering approximation space in Bonikowski et al. [4]. Various pairs of lower and upper approximations in the covering approximation space $(U, \mathcal{C})$ were subsequently studies in [25,26,31,32,34]. For example, Zhu [31,32] has proposed two types of covering-based rough sets. The lower approximations are the same for them and come from Bonikowski et al. [4]. The common lower approximation (also in [25,34])

$$\mathcal{C}_\sharp : X \mapsto \mathcal{C}_\sharp(X) = \cup\{K \in \mathcal{C} : K \subseteq X\}. \tag{1.1}$$

is the same as the $\mathcal{C}$-lower approximation in [2], as will be shown later. Extensive research on covering-based rough sets can be found in [12,15,16,19,24,27,29,30,33].

In the approximation space $(U, \beta)$, the lower and upper approximations as mappings between subsets of $U$ are in fact dual. The upper approximation is a topological closure on $U$, namely, the upper approximation satisfies the Kuratowski closure axioms which define a topological structure on the set $U$. As it is well-known [6], a neighborhood system can be defined based on the so-called topological neighborhood axioms such that this neighborhood system induces the given topological closure. The converse process is also possible except that it may lead to an equivalent neighborhood system but not the original. Various equivalent ways can be used to define this structure. Consequently, it allows us to study the topological structure from different viewpoints.

According to Day [6], topological closure and neighborhood system seem to be the most important, convenient, and simplest topological notions. Generalized topological closure and neighborhood system have been applied in many other disciplines such as chemical organization theory [3], structure analysis [5], biochemistry [21] and biology [22].

In this paper, we study the approximation theory via total pure reflexive (TPR) neighborhood systems, a special kind of neighborhood systems [23]. In the framework of TPR neighborhood systems, the definition of pre-topological equivalence is formulated in terms of neighborhoods; we show that the so-called lower and upper approximations are pre-topological invariants and dual to each other. This is one of the advantages in the study of approximation theory by using TPR neighborhood systems. We construct the corresponding generalized topological closure via the upper approximation. We regard a given covering $\mathcal{C}$ of the set $U$ as a spe-

* Corresponding author. Tel.: +1 989 774 3597; fax: +1 989 774 2414.
E-mail address: enbing.lin@cmich.edu (E.-B. Lin).

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cial form of a TPR neighborhood system $CN : U \to 2^{2^U}$ by taking $CN(x) = \{K \in \mathcal{C} | x \in K\}$ for each $x \in U$. Based on the induced neighborhood system, we define a pair of lower and upper approximations which treats each $CN(x)$ as a local base. In the case of a finite covering, we also discuss the construction of lower and upper approximations by treating each $CN(x)$ as a local subbase. The approximation theory for the first pair is the same as the covering neighborhood system $CN : U \to 2^{2^U}$. The upper approximation of the second pair is a preclosure on $U$, and the family of complements of its fixed points not only forms a topology, but also an Alexandroff topology [1].

This paper is organized as follows. We provide with some preliminary backgrounds, including TPR neighborhood systems, in Section 2. Some salient features of TPR neighborhood systems include the relationships with lower and upper approximations, pre-topological invariant, local bases as well as maximal neighborhood systems which are described in Section 3 as well. As a special kind of TPR neighborhood systems, covering approximation spaces are considered in Section 4. We derive a number of descriptions of covering neighborhood systems in terms of local bases, local subbases and covering approximations. We present examples in Section 3 and Section 4 to illustrate our results. We conclude with several remarks in Section 5.

## 2. Preliminary

Let $U$ be a nonempty set (may be finite or infinite) of all the objects under consideration, referred as the universe (of discourse). The power set of $U$, denoted by $2^U$, is the collection of all subsets of $U$, i.e., $2^U = \{S | S \subseteq U\}$.

We use the symbol "$\subseteq$" for set inclusion. If $A \subseteq B$ and $A \neq B$, we write $A \subset B$.

A collection $\mathcal{C} = \{K_i \subseteq U | i \in I\}$, where $I$ is an index set (may be finite or infinite), of nonempty subsets of $U$ whose union is the whole set $U$ is called a covering of $U$.

By a finite covering we mean a covering with finite members. We recall

**Definition 1.** [4] Let $\mathcal{C}$ be a finite covering of $U$. For $x \in U$, the family

$$\mathrm{Md}(x) = \{K \in \mathcal{C} | x \in K \wedge (\forall S \in \mathcal{C},\ x \in S \wedge S \subseteq K \Rightarrow K = S)\} \quad (2.2)$$

is called the minimal description of the object $x$.

**Lemma 1.** [4] *Let $\mathcal{C}$ be a finite covering of $U$. For $x \in U$, we have*

$$\bigcap_{K \in \mathrm{Md}(x)} K = \bigcap \{K \in \mathcal{C} | x \in K\}. \quad (2.3)$$

### 2.1. $\mathcal{C}$-lower and $\mathcal{C}$-upper approximations

Let $\beta = \{E_i \subseteq U | i \in I\}$, where $I$ is an index set, be a partition of $U$. For any $X \subseteq U$, according to Pawlak [18], Pawlak's lower and upper approximations of $X$, $\beta_*(X)$ and $\beta^*(X)$, respectively, are defined as follows:

$$\beta_*(X) = \cup\{E_i \in \beta | E_i \subseteq X\}, \quad \beta^*(X) = \cup\{E_i \in \beta | E_i \cap X \neq \emptyset\}. \quad (2.4)$$

Equivalently, $\beta_*(X)$ and $\beta^*(X)$ can also be presented as follows [2]:

$$\beta_*(X) = \{x \in U | \exists\ E_i \in \beta,\ \text{such that } x \in E_i\ \&\ E_i \subseteq X\} \quad (2.5)$$
$$\beta^*(X) = \{x \in U | \forall\ E_i \in \beta,\ \text{if } x \in E_i,\ \text{then } E_i \cap X \neq \emptyset\}. \quad (2.6)$$

As it is known, the universe $U$ was assumed to be finite in Pawlak's rough set model. However, Barot and Lin [2] allowed $U$ to be infinite, and they directly generalized Eqs. (2.5) and (2.6) to a covering $\mathcal{C}$ by interpreting $E_i$ as members of $\mathcal{C}$.

**Definition 2.** [2] Let $\mathcal{C}$ be a covering of $U$. For any $X \subseteq U$, the $\mathcal{C}$-lower and $\mathcal{C}$-upper approximations of $X$, $\mathcal{C}_*(X)$ and $\mathcal{C}^*(X)$, respectively, are defined as follows:

$$\mathcal{C}_*(X) = \{x \in U | \exists\ K \in \mathcal{C},\ \text{such that } x \in K\ \&\ K \subseteq X\} \quad (2.7)$$
$$\mathcal{C}^*(X) = \{x \in U | \forall\ K \in \mathcal{C},\ \text{if } x \in K,\ \text{then } K \cap X \neq \emptyset\}. \quad (2.8)$$

### 2.2. The notion of topological spaces

A family $\mathcal{T}$ of subsets of $U$ is called a topology on $U$ if it contains the whole set $U$ and the empty set $\emptyset$, and is closed under finite intersections and arbitrary unions. The pair $(U, \mathcal{T})$ is called a topological space. The members of $\mathcal{T}$ are called $\mathcal{T}$-open, or open sets. We also call $U$ a topological space, with the understanding that $U$ is equipped with a family $\mathcal{T}$.

The complement of an open set is called a closed set. Using de Morgan's laws, topology can be alternatively described in terms of closed sets as well; more precisely, $U$ and $\emptyset$ are closed, arbitrary intersections of closed sets are closed, and finite unions of closed sets are closed.

There are many other ways to set up axioms, including the so-called Kuratowski closure axioms (in short, KC-axioms) and the topological neighborhood axioms (in short, TN-axioms), that can be used to define this structure.

#### 2.2.1. Kuratowski closure axioms

Recall that [7], the closure ($\mathcal{T}$-closure) of a subset $X$, denoted by $c(X)$, of a topological space $(U, \mathcal{T})$ is the intersection of all closed sets containing $X$. Viewing closure as a mapping $c$ between subsets of $U$, it satisfies the following five statements (known as the Kuratowski closure axioms):

1. $c(\emptyset) = \emptyset$.
2. $X \subseteq c(X)$ for all $X \subseteq U$ (extensivity).
3. $c(X \cup Y) \supseteq c(X) \cup c(Y)$ for all nonempty $X, Y \subseteq U$ or equivalently: $X \subset Y$ implies $c(X) \subseteq c(Y)$ for all nonempty $X,\ Y \subseteq U$ (monotonicity).
4. $c(X \cup Y) \subseteq c(X) \cup c(Y)$ for nonempty $X,\ Y \subseteq U$ (subadditivity).
5. $c(c(X)) = c(X)$ for all $X \subseteq U$ (idempotence).

If the axiom of idempotence is relaxed, then the axioms define a preclosure operator.

#### 2.2.2. Topological neighborhood axioms

By a neighborhood system on $U$ we mean a mapping $NS : U \to 2^{2^U}$ which is defined by assigning to each $x$ of $U$ a nonempty collection $NS(x)$ of subsets of $U$. Such a nonempty collection $NS(x)$ and each of its members is called a neighborhood system at $x$ and a neighborhood of $x$, respectively.

A neighborhood system $NS : U \to 2^{2^U}$ will be called a topological neighborhood system, or a topology, on $U$ if for each $x \in U$, $NS(x)$ satisfies the following topological neighborhood axioms [7,13]:

(TN 1) If $N \in NS(x)$, then $x \in N$.
(TN 2) If $N,\ M$ are members of $NS(x)$, then $N \cap M \in NS(x)$.
(TN 3) Superset condition: If $M \supseteq N$ for a nonempty $N \in NS(x)$, then $M \in NS(x)$.
(TN 4) If $N \in NS(x)$, then there is a member $M$ of $NS(x)$ such that $M \subseteq N$ and $M \in NS(y)$ for each $y \in M$ (that is, $M$ is a neighborhood of each of its points).

Omitting (TN 4) leads to a pretopological neighborhood system on $U$.