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Weak transitivity of interval-valued fuzzy relations

Yejun Xu [a,b], Huimin Wang [a,b,*], Dejian Yu [c]

[a] State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Hohai University, Nanjing, Jiangsu 210098, PR China
[b] Business School, Hohai University, Nanjing, Jiangsu 211100, PR China
[c] School of Information, Zhejiang University of Finance & Economics, 18 Xueyuan Street, Hangzhou 310018, PR China

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ABSTRACT

In this paper, we define and study the weak transitivity of interval-valued fuzzy relations (IVFRs). We propose the weak transitivity index (*WTI*) to measure the transitivity consistency degree of an IVFR, which is to count the cycles of length 3 in the digraph. Afterwards, an algorithm is proposed to compute the *WTI* and to locate each cycle, as well as to find the inconsistent judgments in an IVFR. In order to resolve the intransitivities of an IVFR, another algorithm is developed to find and remove all the 3-cycles in the digraph. Our method can not only repair the weak intransitivity for an IVFR, but also preserve the initial preference information as much as possible. Finally, two examples are shown to illustrate the proposed method.

1. Introduction

Interval-valued fuzzy set (IVFS) theory [23] is an extension of fuzzy theory. The membership degree of each element on an IVFS is defined on a closed subinterval of [0,1]. IVFSs have been used in a number of different fields: image processing [34], interval-valued logic [35,36], approximate reasoning [4,7,23] and so on.

Transitivity is a fundamental notion in decision theory. It is most universally assumed in disciplines of decision theory and generally accepted in a principle of rationality. Yet, it is often violated in actual choice, particularly in pairwise choices. A first task for decision science is thus the resolution of intransitivities [27]. The weak transitivity is the usual transitivity condition that a logical and consistent person should use if he/she does not want to express inconsistent opinions, and therefore it is the minimum requirement condition that a consistent fuzzy preference relation should verify [31]. Weak transitivity is in fact acyclic about the alternatives ranking, i.e., if an alternative *A* is preferred or equivalent to *B*, and *B* is preferred or equivalent to *C*, then *A* must be preferred or equivalent to *C*. The transitivity assumption can be used to check for the judgment consistency of a decision maker (DM). If a DM provides a preference relation does not possess transitivity (i.e., inconsistency problems exist), the ranking result of alternatives is misleading [25,26,29].

Transitivity of a fuzzy preference relation has been received greatly attention in the past decades [2,3,9–16,18,20,28,29], such as weak transitivity (or called weak stochastic transitivity) [8,15,16,18,31,32,40,41], max–min transitivity [19,25,31,33,37], max-max transitivity [25,31], restricted max–min transitivity (or moderate stochastic transitivity) [8,15,16,25,31–33,37], restricted max-max transitivity (or strong stochastic transitivity) [8,15,16,18,25,31,32], multiplicative consistency [8,25] and additive consistency [8,25]. It should be pointed out that, strictly speaking, additive consistency is not a type of transitivity [17]. Gonzales-del-Campo et al. [22] proposed an algorithm to compute the transitive closure for an IVFR. IVFRs are also common fuzzy relations which experts express their comparison information for alternatives. The comparison information are not exact numerical values but interval numbers. Thus, the transitivity is also an important problem for IVFRs. However, little work has been done on the transitivity problem of IVFRs. Therefore, it is important to pay attention to this problem. This is the subject of the present paper.

In this paper, we give a definition of the weak transitivity for IVFRs. We propose the weak transitivity index (*WTI*) to measure the consistency degree of an IVFR, which is to count the 3-cyles in the digraph. A procedure is proposed to compute the *WTI* and locate each cycle, as well as to find the inconsistent judgments in an IVFR. If an IVFR is not weakly transitive, another algorithm is developed to find and remove all the 3-cycles in the digraph. Moreover, our improving method can preserve the initial preference information as much as possible.

The rest of the paper is set out as follows. Section 2 gives the basic concepts related to IVFSs. We give the definition of weak

* Corresponding author at: State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Hohai University, Nanjing, Jiangsu 210098, PR China.
E-mail addresses: xuyejohn@163.com (Y. Xu), hmwang@hhu.edu.cn (H. Wang), yudejian62@126.com (D. Yu).

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transitivity for an IVFR. Section 3 proposes a procedure to judge whether an IVFR is weakly transitive, as well as to find the inconsistent judgments in an IVFR. Section 4 develops a method to repairing the intransitivities for an IVFR. Section 5, two examples are illustrated to show the effectiveness and validity of the proposed methods. The conclusions, some characteristics and advantages of the proposed methods and future research are presented in Section 6.

2. Preliminaries

In the following, we introduce some basic concepts related to interval-valued fuzzy sets.

Definition 1. Let Y be a universe of discourse, then a fuzzy set:

$$A = \{\langle y, \mu_A(y)\rangle | y \in Y\} \tag{1}$$

defined by Zadeh [45] is characterized by a membership function: μ_A:$Y \to [0,1]$, where $\mu_A(y)$ denotes the degree of membership of the element y to the set A.

We will denote with $L([0,1])$ the set of all closed subintervals of the closed interval $[0,1]$. That is,

$$L([0,1]) = \{\mathbf{x} = [x^-, x^+] | (x^-, x^+) \in [0,1]^2 \text{ and } x^- \leqslant x^+\}$$

Definition 2 [5]. An interval-valued fuzzy set (IVFS) A, on the universe $U \neq \emptyset$, is a set such that

$$A = \{(u, A(u) = [A^-(u), A^+(u)]) | u \in U\} \tag{2}$$

where the function A: $U \to L([0,1])$ is called the membership function.

For convenience, we call $\alpha = [\alpha^-, \alpha^+]$ an interval-valued fuzzy value (IVFV), where $\alpha^- \in [0,1]$, $\alpha^+ \in [0,1]$, $\alpha^- \leqslant \alpha^+$. Based on the concepts of score function and accuracy degree of intuitionistic fuzzy values, in the following, we define the corresponding concepts for IVFVs, which are used to compare two IVFVs.

Definition 3. Let $\alpha = [\alpha^-, \alpha^+]$ be an IVFV, where $\alpha^- \in [0,1]$, $\alpha^+ \in [0,1]$, $\alpha^- \leqslant \alpha^+$. The score of α can be evaluated by the score function s shown as

$$s(\alpha) = \alpha^- + \alpha^+ - 1 \tag{3}$$

where $s(\alpha) \in [-1,1]$. The larger the score $s(\alpha)$, the greater the IVFV α.

An accuracy function h to evaluate the degree of accuracy of α can be expressed as:

$$h(\alpha) = \alpha^- + 1 - \alpha^+ \tag{4}$$

where $h(\alpha) \in [0,1]$. The lager the value of $h(\alpha)$, the more the degree of accuracy of α.

Definition 4. Let $\alpha = [\alpha^-, \alpha^+]$, $\beta = [\beta^-, \beta^+]$ be two IVFVs, $s(\alpha) = \alpha^- + \alpha^+ - 1$ and $s(\beta) = \beta^- + \beta^+ - 1$ be the scores of α and β, respectively, and let $h(\alpha) = \alpha^- + 1 - \alpha^+$ and $h(\beta) = \beta^- + 1 - \beta^+$ be the accuracy degrees of α and β, respectively, then

- If $s(\alpha) < s(\beta)$, then α is smaller than β, denoted by $\alpha < \beta$.
- If $s(\alpha) = s(\beta)$, then

(1) If $h(\alpha) = h(\beta)$, then α and β represent the same information, denoted by $\alpha = \beta$.
(2) If $h(\alpha) < h(\beta)$, then α is smaller than β, denoted by $\alpha < \beta$.

For a decision making problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of alternatives. In the process of decision making, a DM generally needs to provide his/her preferences for each pair of alternatives, and perhaps it is possible that he/she is not so sure about it. Thus, it is very suitable to express the DM's preference values with IVFVs rather than exact numerical values, and then constructs an interval-valued fuzzy relation (IVFR), which can be defined as follows.

Definition 5 [43]. An IVFR R on the set X is represented by a matrix $R = (r_{ij})_{n\times n} \subset X \times X$, where

$$r_{ij} = \left[r_{ij}^-,\ r_{ij}^+\right],\ r_{ji} = \left[r_{ji}^-,\ r_{ji}^+\right],\ r_{ij}^- + r_{ji}^+ = r_{ij}^+ + r_{ji}^- = 1,\ r_{ij}^+ \geqslant r_{ij}^- \geqslant 0,$$
$$r_{ii} = [0.5, 0.5], \text{ for all } i,j = 1,2,\ldots,n \tag{5}$$

r_{ij} is interpreted as the preference degree of the alternative x_i over x_j: (1) $r_{ij} = [0.5,0.5]$ (i.e. $r_{ij}^- = r_{ij}^+ = 0.5$) denotes indifference between x_i and x_j ($x_i \sim x_j$); (2) $[0.5,0.5] < r_{ij} \leqslant [1,1]$ denotes x_i is strictly preferred to $x_j (x_i \succ x_j)$. Especially, $r_{ij} = [1,1]$ denotes that x_i is definitely preferred to x_j; (3) $[0,0] \leqslant r_{ij} < [0.5,0.5]$ denotes that x_j is strictly preferred to $x_i (x_j \succ x_i)$. Especially, $r_{ij} = [0,0]$ denotes that x_j is definitely preferred to x_i.

In the following, we define the weak transitivity for IVFRs.

Definition 6. Let $R = (r_{ij})_{n\times n}$ be an IVFR, where $r_{ij} = \left[r_{ij}^-, r_{ij}^+\right]$, $i,j = 1,2,\ldots,n$, for all i, j, $k = 1, 2, \ldots, n$, $i \neq j \neq k$

(1) if $r_{ik} > [0.5,0.5]$ and $r_{kj} \geqslant [0.5,0.5]$, we have $r_{ij} > [0.5,0.5]$;
(2) if $r_{ik} \geqslant [0.5,0.5]$ and $r_{kj} > [0.5,0.5]$, we have $r_{ij} > [0.5,0.5]$;
(3) if $r_{ik} = [0.5,0.5]$ and $r_{kj} = [0.5,0.5]$, we have $r_{ij} = [0.5,0.5]$.

then R is weakly transitive.

Definition 7. Let $R = (r_{ij})_{n\times n}$ be an IVFR, where $r_{ij} = \left[r_{ij}^-, r_{ij}^+\right]$, $i,j = 1,2,\ldots,n$, for all i, j, $k = 1, 2, \ldots, n$, $i \neq j \neq k$, if $r_{ik} > [0.5,0.5]$, $r_{kj} > [0.5,0.5]$, we have $r_{ij} > [0.5,0.5]$, then R is strict weakly transitive.

In the following, we will discuss the weak transitivity of the IVFR from the graph theory point of view. Some basic theory of digraph is presented as follows.

Definition 8. Let $R = (r_{ij})_{n\times n}$ be an IVFR, where $r_{ij} = \left[r_{ij}^-, r_{ij}^+\right]$, $i,j = 1,2,\ldots,n$, we define the adjacency matrix $E = (e_{ij})_{n\times n}$ of R as follows:

$$e_{ij} = \begin{cases} 1, & r_{ij} \geqslant [0.5,0.5], i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Let $R = (r_{ij})_{n\times n}$ be an IVFR, we can construct the digraph $G = (V,A)$ of R, where $V = \{v_1, v_2, \ldots, v_n\}$ denotes the node set, $A = \{(v_i, v_j) | i \neq j, r_{ij} \geqslant [0.5,0.5]\}$ denotes the arc set. That is, if $i \neq j$, $r_{ij} > [0.5,0.5]$, then there is a directed arc in G from v_i to v_j, it is denoted by (v_i, v_j) or $v_i \to v_j$. r_{ij} is called the weight of the arc (v_i, v_j). Therefore, if $r_{ij} = [0.5,0.5]$ $(i \neq j)$, then there is an arc from v_i to v_j, and also an arc from v_j to v_i. A directed path ρ in a graph G is a sequence of arcs $v_{i_1}, v_{i_2}, v_{i_3}, \ldots$.. in G, where the nodes v_{i_k} are different. The length of a directed path is the number of successive arcs in the directed path. A cycle is a directed path that begins and ends at the same node.

Proposition 1. *Let $R = (r_{ij})_{n\times n}$ be an IVFR, then there exists a directed path ρ of length $n-1$ in the digraph G of R.*