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Consensus measures constructed from aggregation functions and fuzzy implications

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ABSTRACT

We focus on the problem of constructing functions that are able to measure the degree of consensus for a set of inputs provided over the unit interval. When making evaluations based on inputs from multiple criteria, sources or experts, the resulting output can be seen as the value which best represents the individual contributions. However it may also be desirable to know the extent to which the inputs agree. Does the representative value reflect a universal opinion? Or has there been a high degree of tradeoff? We consider the properties relating to such consensus measures and propose two general models built component-wise from aggregation functions and fuzzy implications.

1. Introduction

Aggregation functions such as the mean and median are used in a wide range of decision making contexts to summarize a set of inputs with a single output. They can be used to provide an overall rating for an item or candidate based on multiple criteria, or to combine the preferences of experts into a single group evaluation. In some situations, it may also be informative to have an idea of whether the inputs agree with one another, or whether the aggregated score is the result of a compromise between a number of disparate sub-groups.

For instance, suppose some friends provide ratings for three of the *X-men* films (shown in Table 1).

As well as comparing the average evaluations, we can also see that whilst everyone more or less agrees that *X-men III* was not very good, and *X-Men: First Class* was not too bad, there is a lack of consensus regarding *Wolverine*.

Consensus measures, that is, functions which give an overall idea of how much the inputs agree with one another, have been employed increasingly in decision making contexts. Such measures have been used in voting and preferences aggregation [1,2,12,20–23,31,32], for example to describe a set of voters and group them according to the similarity in their preferences. As with the standard deviation alongside the mean in statistical summaries, consensus measures can provide an indication of reliability or the degree to which an overall evaluation reflects the opinions of a group. As such, they have also been used to inform consensus reaching processes [3,10,16–18,25–29,38,39,41,40,42,43], where a minimum level of consensus can be set and a final decision may not be accepted if the consensus measure output is below this threshold. The consensus level between the pairwise preferences of an individual and the group can also be used to make recommendations that will increase the overall agreement between experts. The work of Tastle et al. has also looked at consensus for a set of opinions expressed over Likert scales [36].

In this article, we consider the problem of measuring the degree of consensus given a set of inputs expressed over the unit interval. We focus on how to formally state the consensus properties considered in [1,2,20,36] when applied to this setting and then propose some generalized operators that satisfy these properties with a suitable choice of components. While our considerations are contained to real inputs between 0 and 1, our investigations lay the framework for extensions to other environments such as lattice inputs. This work continues that begun in [7], where we first proposed one of the consensus models considered here and compared it with some existing functions in terms of the consensus properties described in [36].

The paper will be structured as follows. We will firstly outline the necessary background in aggregation functions and fuzzy implications in the Preliminaries section. We include the definition of the Bonferroni mean, the construction of which motivates the proposed consensus measure. In Section 3, we will bring together the properties relating to consensus in other settings and adapt their definitions to the case of inputs expressed over the unit interval. We will include reference to some existing consensus measures with respect to their satisfaction of the given properties. In Section 4, we will propose our consensus operators constructed

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Table 1
Overall evaluations based on individual ratings.

Film	Trevor	Bailey	Natasha	Josh	Average
X-men III	0.3	0.2	0.2	0.4	0.275
Wolverine	0.2	0.3	0.9	0.8	0.55
X-Men: First Class	0.65	0.7	0.65	0.6	0.65

from aggregation functions and fuzzy implications. The first of these is based on aggregating the fuzzy implications of each pair of inputs, while the second looks at implications between each input and the average of the remaining inputs. We will discuss how the choice of components affects the satisfaction of consensus properties defined in the preceding section. We will provide some final thoughts in the Conclusion.

2. Preliminaries

Our proposed consensus measure is composed of aggregation functions and fuzzy implications. We will give an overview of the necessary background on these topics in this section.

2.1. Aggregation functions

The theory of aggregation functions has received considerable attention in recent years, with many results in terms of understanding their properties and potential applications. Recent monographs dedicated to the topic give an overview of the state-of-the-art [9,24,37]. Aggregation functions are used to merge inputs into a single output that is seen to be representative, subject to given properties. In group decision making, aggregation functions are used to combine the evaluations of the experts into an overall evaluation for each item or object under consideration. Inputs and outputs can be considered over different domains, however we will contain our considerations to the case where both are expressed over the unit interval.

Definition 1 (*Aggregation function*). [9,24] A function $f:[0,1]^n \to [0,1]$, $n > 1$ is called an aggregation function if it is monotone non-decreasing in each argument and satisfies the boundary conditions $f(0,\dots,0) = 0$ and $f(1,\dots,1) = 1$.

Aggregation functions are considered to be *averaging* when $\min(\mathbf{x}) \leqslant f(\mathbf{x}) \leqslant \max(\mathbf{x})$, *conjunctive* when $f(\mathbf{x}) \leqslant \min(\mathbf{x})$, *disjunctive* when $\max(\mathbf{x}) \leqslant f(\mathbf{x})$, and *mixed* otherwise. We note that averaging aggregation functions are *idempotent*, i.e. $f(t,t,\dots,t) = t$, with well known examples including the arithmetic mean (also sometimes just referred to as the average) and the median. T-norms and t-conorms are the archetypal examples of conjunctive and disjunctive functions respectively, generalizing the AND and OR operations of 2-valued logic. An aggregation function f is said to by symmetric when for all permutations $\pi(i)$ on $\{1,\dots,n\}$ and $\mathbf{x} \in [0,1]^n$ it holds that $f(x_1,x_2,\dots,x_n) = f(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)})$.

We provide the definition of weighted quasi-arithmetic means, which generalize a number of well known averaging aggregation functions.

Definition 2 (*Weighted quasi-arithmetic mean*). For a strictly monotone continuous generating function $g:[0,1] \to [-\infty,\infty]$ and weighting vector $\mathbf{w}$, the weighted quasi-arithmetic mean is given by

$$QAM_{\mathbf{w}}(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right). \tag{1}$$

Special cases include weighted arithmetic means, $WAM(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$ where $g(t) = t$, weighted power means $PM_q(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i x_i^q\right)^{1/q}$, where $g(t) = t^q$ and weighted geometric means $G(\mathbf{x}) = \prod_{i=1}^{n} x_i^{w_i}$ if $g(t) = -\ln t$. The weights w_i are usually seen to indicate the relative importance of a given input source, are non-negative and sum to one.

The consensus measures we will propose are based on the form of the Bonferroni mean. It was defined in 1950 [11] and has been generalized by Yager [44] and others [34,45,8].

Definition 3 (*Bonferroni mean*). Let $p, q \geqslant 0$ and $x_i \geqslant 0$, $i = 1, \dots, n$. The Bonferroni mean is the function

$$B^{p,q}(\mathbf{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^p x_j^q\right)^{\frac{1}{p+q}}. \tag{2}$$

By rearranging the terms, the Bonferroni mean can also be expressed,

$$B^{p,q}(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p \left(\frac{1}{n-1} \sum_{j=1, j\neq i}^{n} x_j^q\right)\right)^{\frac{1}{p+q}}. \tag{3}$$

In both equations, we have an arithmetic mean which lies inside the $(p+q)$th root. For Eq. (2), the arguments of this mean are the product pairs $x_i^p x_j^q$, while in Eq. (3) we take the product of each component x_i with the arithmetic mean of the inputs when x_i is excluded. The generalized form of the Bonferroni mean presented in [8] allows for the mean operations to be replaced with any averaging functions and the product operation with any 2-variate function whose diagonal $f(t,t)$ is invertible.

We will give an overview of fuzzy implications in the following subsection.

2.2. Fuzzy Implications

Fuzzy implications generalize the classical implication operator $\to$ in two-valued logic. Their properties and applications in fuzzy systems have also received considerable attention in recent years (see the monograph [4], edited book [5] and the research papers [15,6,33] for more detailed background on state-of-the-art results and applications). We will adopt the following definition.

Definition 4 (*Fuzzy implication*). [4,33] A function $I:[0,1]^2 \to [0,1]$ is called a fuzzy implication (or implication function) if it is monotone non-increasing in the first argument, monotone non-decreasing in the second argument and satisfies the boundary conditions, $I(0,0) = 1$, $I(1,1) = 1$ and $I(1,0) = 0$.

Implication functions can be defined from aggregation functions and negations. We define a negation as follows.

Definition 5 (*Fuzzy negation*). [9,4] A function $N:[0,1] \to [0,1]$ is called a fuzzy negation (or negation function) if it is monotone non-increasing and satisfies the boundary conditions $N(0) = 1$, $N(1) = 0$.

A fuzzy negation is said to be *strict* if is continuous and strictly monotone decreasing. A *strong negation* is strict and involutive, i.e. $N(N(t)) = t$.

Negations play an important role in the study of fuzzy implications as they can be used to consider properties such as contraposition and define various classes. For example, the (S,N)-implications are defined with respect to a t-conorm S and a negation N, and given by $I(x,y) = S(N(x),y)$, while another class, (Q,L)-implications, are defined with respect to a negation N, a t-norm T and t-conorm S as $I(x,y) = T(N(x),S(x,y))$.