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# Distance and similarity measures for higher order hesitant fuzzy sets

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ABSTRACT

In this study, we extend the hesitant fuzzy set (HFS) to its higher order type and refer to it as the higher order hesitant fuzzy set (HOHFS). HOHFS is the actual extension of HFS that enables us to define the membership of a given element in terms of several possible generalized type of fuzzy sets (G-Type FSs). The rationale behind HOHFS can be seen in the case that the decision makers are not satisfied by providing exact values for the membership degrees and therefore the HFS is not applicable. However, in order to indicate HOHFSs have a good performance in decision making, we first introduce some information measures for HOHFSs and then apply them to multiple attribute decision making with higher order hesitant fuzzy information.



## 1. Introduction

As a topic for further research, the theory of generalizing and extending fuzzy sets (FSs) [30] seems very prospective.

Nowadays, many different extensions of FSs are known: L-fuzzy sets (L-FSs) [10], interval-valued fuzzy sets (IVFSs) [22], vague sets (VSs) [4], intuitionistic fuzzy sets (IFSs) [2], interval-valued intuitionistic fuzzy sets (IVIFSs) [3], linguistic fuzzy sets (LFSs) [26], type-2 fuzzy sets (T2FSs) [16], type-$n$ fuzzy sets (TnFSs) [7] and fuzzy multisets (FMSs) [17].

An interesting extension has been given by Torra and Narukawa [20,21] who introduced hesitant fuzzy sets (HFSs) in which the membership is the union of several memberships of FSs. HFSs are quite suit for the situation where we have a set of possible values, rather than a margin of error (as in IFSs) or some possibility distribution on the possible values (as in T2FSs). Later, a number of other extensions of the HFSs have been developed such as dual hesitant fuzzy sets (DHFSs) [31], generalized hesitant fuzzy sets (G-HFSs) [18] and hesitant fuzzy linguistic term sets (HFLTSs) [19].

However, HFS [20] and its recent generalization G-HFS [18] have their inherent drawbacks, because they express the membership degrees of an element to a given set only by crisp numbers or IFSs. In many practical decision making problems, the information provided by a decision maker might often be described by FSs (instead of crisp numbers) or other FS extensions (instead of IFSs). Therefore, it is difficult for the decision makers to provide exact crisp values or just IFSs for the membership degrees. This difficulty can be avoided using a higher order HFS (HOHFS) introduced later for the membership degrees. The HOHFS is fit for the case when the decision makers have a hesitation among several possible memberships with uncertainties. The HOHFS is the actual extension of HFS encompassing not only FSs, IFSs, T2FSs and HFSs, but also the recent extension of HFSs, called G-HFSs.

A growing number of studies focus on the distance measure and the similarity measure for HFSs [28] and some extensions of HFS [9,8]. Distance measures are fundamentally important in various fields such as decision making, market prediction, and pattern recognition.

In view of the theorems [9] demonstrating that the similarity measure and the distance measure can be transformed by each other, this article deals mainly with distance measure for HOHFSs. It is worth mentioning that the existing information measures of HFSs have up to now proposed under restricted assumptions [9,8,18,28,31] that are (A1) all the elements in each HFE are rearranged in increasing (or decreasing) order and (A2) the number of values in different HFEs must be indifferent. Here, instead of making such assumptions about the form of HOHFSs, the proposed information measures are able to address the discrimination of HOHFSs.

The present paper is organized as follows: An extension of HFSs, which is referred to as the higher order HFS (HOHFS) introduced in Section 2. Section 3 presents the axioms for distance and similarity measures, and gives a variety of distance measures for HOHFSs. In Section 4, we apply the proposed distance measures for HOHFSs to multi-attribute decision-making with the known weight information on attributes. Finally, conclusion is drown in Section 5.

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## 2. Preliminaries

This section is devoted to describing the basic definitions and notions of fuzzy set (FS) and its new generalization which are referred to as the higher order hesitant fuzzy set (HOHFS). Actually, the HOHFS is a generalization of hesitant fuzzy set (HFS), originally introduced by Torra and Narukawa [20,21].

An ordinary fuzzy set (FS) $A$ in $X$ is defined [30] as $A = \{\langle x, A(x)\rangle : x \in X\}$, where $A: X \to [0,1]$ and the real value $A(x)$ represents the degree of membership of $x$ in $A$.

**Definition 2.1.** Let $X$ be the universe of discourse. A generalized type of fuzzy set (G-Type FS) on $X$ is defined as

$$\widetilde{A} = \{\langle x, \widetilde{A}(x)\rangle : x \in X\}, \tag{1}$$

where

$$\widetilde{A} : X \to \psi([0,1]).$$

Here, $\psi([0,1])$ denotes a family of crisp or fuzzy sets that can be defined with in the universal set $[0,1]$.

It is noteworthy that most of the existing extensions of ordinary FS are special cases of G-Type FS, for instance, [13].

- if $\psi([0,1]) = [0,1]$, then the G-Type FS $\widetilde{A}$ reduces to an ordinary FS;
- if $\psi([0,1]) = \varepsilon([0,1])$ denoting the set of all closed intervals, then the G-Type FS $\widetilde{A}$ reduces to an IVFS;
- if $\psi([0,1]) = \mathcal{F}([0,1])$ denoting the set of all ordinary FSs, then the G-Type FS $\widetilde{A}$ reduces to a T2FS;
- if $\psi([0,1]) = L$ denoting a partially ordered Lattice, then the G-Type FS $\widetilde{A}$ reduces to a L-FS.

**Definition 2.2.** [20,21] Let $X$ be the universe of discourse. A hesitant fuzzy set (HFS) on $X$ is symbolized by

$$H = \{\langle x, h(x)\rangle : x \in X\},$$

where $h(x)$, referred to as the hesitant fuzzy element (HFE) [25], is a set of some values in $[0,1]$ denoting the possible membership degree of the element $x \in X$ to the set $H$.

**Example 2.1.** If $X = \{x_1, x_2, x_3\}$ is the universe of discourse, $h(x_1) = \{0.2, 0.4, 0.5\}$, $h(x_2) = \{0.3, 0.4\}$ and $h(x_3) = \{0.3, 0.2, 0.5, 0.6\}$ are the HFEs of $x_i$ ($i = 1,2,3$) to a set $H$, respectively. Then $H$ can be considered as a HFS, i.e.,

$$H = \{\langle x_1, \{0.2, 0.4, 0.5\}\rangle, \langle x_2, \{0.3, 0.4\}\rangle, \langle x_3, \{0.3, 0.2, 0.5.0.6\}\rangle\}.$$

As can be seen from Definition 2.2, HFS expresses the membership degrees of an element to a given set only by several real numbers between 0 and 1, while in many real-world situations assigning exact values to the membership degrees does not describe properly the imprecise or uncertain decision information. Thus, it seems to be difficult for the decision makers to rely on HFSs for expressing uncertainty of an element.

To overcome the difficulty associated with expressing uncertainty of an element to a given set, the concept of higher order hesitant fuzzy set (HOHFS) is introduced here to let the membership degrees of an element to a given set be expressed by several possible G-Type FSs.

**Definition 2.3.** Let $X$ be the universe of discourse. A higher order hesitant fuzzy set (HOHFS) on $X$ is defined in terms of a function that when applied to $X$ returns a set of G-Type FSs. A HOHFS is denoted by

$$\widetilde{H} = \{\langle x, \tilde{h}(x)\rangle : x \in X\}, \tag{2}$$

where $\tilde{h}(x)$, referred to as the higher order hesitant fuzzy element (HOHFE), is a set of some G-Type FSs denoting the possible membership degree of the element $x \in X$ to the set $\widetilde{H}$. In this regards, the HOHFS $\widetilde{H}$ is also represented as

$$\widetilde{H} = \{\langle x, \{\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l_x)}(x)\}\rangle : x \in X\},$$

where all $\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l_x)}(x)$ are G-Type FSs on $X$.

**Example 2.2.** If $X = \{x_1, x_2, x_3\}$ is the universe of discourse,

$$\tilde{h}(x_1) = \{\tilde{h}^{(1)}(x_1) = (0.2, 0.4),\ \ \tilde{h}^{(2)}(x_1) = (0.5, 0.3)\},$$
$$\tilde{h}(x_2) = \{\tilde{h}^{(1)}(x_2) = (0.3, 0.4)\},$$
$$\tilde{h}(x_3) = \{\tilde{h}^{(1)}(x_3) = (0.3, 0.2),\ \ \tilde{h}^{(2)}(x_3) = (0.1, 0.3),\ \ \tilde{h}^{(3)}(x_3) = (0.5, 0.4)\},$$

are the HOHFEs of $x_i$ ($i = 1,2,3$) to a set $\widetilde{H}$, respectively, where G-Type FSs $\tilde{h}^{(k)}(x_i) = (\mu_{ki}, \nu_{ki})$ are intuitionistic fuzzy set (IFS) [2] such that $0 \leqslant \mu_{ki}, \nu_{ki} \leqslant 1$ and $0 \leqslant \mu_{ki} + \nu_{ki} \leqslant 1$ for $k = 1, 2, \ldots, l_{x_i}$ and $i = 1, 2, \ldots, |X| = 3$. Then $\widetilde{H}$ can be considered as a HOHFS, i.e.,

$$\widetilde{H} = \{\langle x_1, \{(0.2, 0.4), (0.5, 0.3)\}\rangle, \langle x_2, \{(0.3, 0.4)\}\rangle,$$
$$\langle x_3, \{(0.3, 0.2), (0.1, 0.3), (0.5, 0.4)\}\rangle\}.$$

Among the generalization of ordinary FS (type-1 FS), the most widely used extensions are the following: type-2 fuzzy sets (T2FSs) whose membership degrees are also fuzzy, that is, instead of being crisp values in $[0,1]$ the membership degrees are themselves FSs; IFSs extends FSs by a hesitancy function, thus the membership takes the form of an interval.

In view of Definition 2.3 and the latter review of some FS extensions, it is easily deduced that each HOHFS becomes an T2FS if all its G-Type FSs are the same. That is, if $\tilde{h}^{(1)}(x) = \cdots = \tilde{h}^{(l_x)}(x) =: \tilde{\bar{h}}(x)$ for any $x \in X$, then the HOHFS $\widetilde{H} = \{\langle x, \tilde{\bar{h}}(x)\rangle : x \in X\}$ reduces to a T2FS.

It is noteworthy that the notions of interval-valued hesitant fuzzy set (IVHFS) [9] and interval type-2 fuzzy set (IT2FS) [14] both are special cases of HOHFSs. A HOHFS $\widetilde{H} = \{\langle x, \tilde{h}(x)\rangle : x \in X\}$ reduces to an IVHFS, when all G-Type FSs $\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l_x)}(x)$ for any $x \in X$ are considered as closed intervals of real numbers in $[0,1]$. Furthermore, an IVHFS $\widetilde{H} = \{\langle x, [\widetilde{h_l}(x), \widetilde{h_r}(x)]\rangle : x \in X\}$ reduces to an IT2FS, when all intervals satisfy $\left[\tilde{h}_l^{(1)}(x), \tilde{h}_r^{(1)}(x)\right] = \cdots = \left[\tilde{h}_l^{(l_x)}(x), \tilde{h}_r^{(l_x)}(x)\right] =: [\tilde{\bar{h}}_l(x), \tilde{\bar{h}}_r(x)]$ for any $x \in X$.

Recently, it is extended HFSs by intuitionistic fuzzy sets (IFSs) and referred to them as generalized hesitant fuzzy sets (G-HFSs) [18]. It is stated in [18] that FSs, IFSs and HFSs are special cases of G-HFSs. Obviously, a G-HFS $\widetilde{H} = \{\langle x, \tilde{h}(x)\rangle : x \in X\}$ is also an special case of HOHFS where all G-Type FSs $\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l_x)}(x)$ for any $x \in X$ are considered as IFSs. This implies that HOHFSs are more useful than G-HFSs to deal with decision making, clustering, pattern recognition, image processing, etc., when experts have a hesitation among several possible memberships for an element to a set.

## 3. Distances and similarities for HOHFSs

There exist many studies deal with the distance measure and the similarity measure for the concepts of FSs [30], IFSs [1] and interval-valued intuitionistic fuzzy sets (IVIFSs) [23]. Little effort has been made as to the similarity measures for T2FSs [15] while there are several similarity measures for interval T2FSs (IT2FSs), for instance, Gorzalczany's degree of compatibility [11], Bustinces