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The aggregation of multiple three-way decision spaces

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ABSTRACT

Based on the theory of three-way decisions proposed by Yao, Hu established three-way decision spaces on fuzzy lattices and partially ordered sets. At the same time, multiple three-way decision spaces and its corresponding three-way decisions were also established. How to choose a method for the transformation from multiple three-way decision spaces to a single three-way decision space? This is one of the main problems on multiple three-way decision spaces. In connection with the transformation question on multiple three-way decision spaces, this paper gives out an aggregation method from multiple three-way decision spaces to a single three-way decision space through an axiomatic complement-preserving aggregation function. These aggregation methods in the partially set [0,1] contain the weighted average three-way decisions, max-min average three-way decisions and median three-way decisions etc. These methods are generalized to three-way decisions over two groups of multiple three-way decision spaces. At last we illustrate aggregation methods of multiple three-way decision spaces through a practical example.

1. Introduction

Since three-way decisions (3WD) were proposed by Yao [37], many authors had studied 3WD [5,16,17,22,38–40]. The existing studies focus mainly on the following four aspects.

- Three-way decisions based on decision-theoretic rough sets are generalized to various fuzzy sets, such as Deng and Yao considered fuzzy sets [5]; Liang and Liu et al. discussed triangular fuzzy sets [18], Liang and Liu looked upon interval-valued fuzzy sets [16] and intuitionistic fuzzy sets [17]; Zhao and Hu also considered interval-valued fuzzy sets [47,48]; Hu analyzed hesitant fuzzy sets and interval-valued hesitant fuzzy sets [8] etc.
- Three-way decisions based on decision-theoretic rough sets are generalized to more patterns, such as Qian and Zhang et al. introduced multigranulation into decision-theoretic rough sets [30]; To reduce boundary regions, Chen and Zhang et al. proposed multi-granular three-way decision based on the multiple-views of granularity [4]; Sang and Liang et al. considered decision-theoretic rough sets under dynamic granulation [31] etc.
- The theoretical frameworks on three-way decisions are studied, such as the domain of evaluation functions [38], construction and interpretation of evaluation functions [37–39], the mode of three-way decisions [39], the theory of three-way decision spaces [7,8,11] and trisecting-and-acting framework of three-way decisions [42] etc.
- The theory of three-way decisions has been applied to incomplete information system [20], risk decision making [15], classification [21] and clustering [43], investment [23], multi-agent [34], group decision making [19], recommender systems [46], face recognition [14] and social networks [26] etc.

For theoretical development of three-way decisions, Hu systematically studied three-way decision models in rough sets and probabilistic rough sets, introduced axiomatic definitions for decision measurement, decision condition and decision evaluation function and established three-way decision spaces based on fuzzy lattices [7,11] and partially ordered sets [8]. The so-called fuzzy lattice is a complete distributive lattice with an involutive negator (i.e. inverse order and involutive mapping). There are numerous popular fuzzy lattices used in classical logic and fuzzy logic such as crisp sets, fuzzy sets [44], shadowed sets [24,25], intuitionistic fuzzy sets [1,2], interval-valued fuzzy sets [45] and interval sets [35,36]. A fuzzy lattice is also a partially ordered set. There are many partially ordered sets, which are not fuzzy lattices, such as hesitant

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fuzzy sets [33], interval-valued hesitant fuzzy sets [3], type-2 fuzzy sets [9] and interval-valued type-2 fuzzy sets [10].

At the same time, based on multi-granulation rough sets [27–31], multiple three-way decision spaces were further discussed in [7]. As a result of the classical single-granulation rough set theory, a multi-granulation rough set model (MGRS) has been developed [28,29] which is a kind of information fusion strategy through fusing multiple granular structures. The following are some existing multi-granulation fusion strategies.

(1) Pessimistic strategy [27,30].
(2) Optimistic strategy [28–30].
(3) Dynamic strategy [31].

In this paper, we consider two problems. The first problem is *are these existing strategies reasonable*? Another one is *are there other reasonable strategies*? This paper answers these problems through considering aggregation methods from multiple three-way decision spaces to a single three-way decision space which is referred to as the aggregation strategy.

From Note 3.1 in [7], we can see that if $E_1, E_2, \ldots, E_n$ are n decision evaluation functions, then $\wedge_{i=1}^n E_i(A)(x)$ and $\vee_{i=1}^n E_i(A)(x)$ are not necessarily decision evaluation functions because they do not meet the third axiom, Complement Axiom. Are there some methods to construct a decision evaluation function from n decision evaluation functions $E_1, E_2, \ldots, E_n$? Although $\wedge_{i=1}^n E_i(A)(x)$ and $\vee_{i=1}^n E_i(A)(x)$ are not decision evaluation functions, $\frac{1}{2}(\wedge_{i=1}^n E_i(A)(x) + \vee_{i=1}^n E_i(A)(x))$ is a decision evaluation function over [0, 1]. And $\frac{1}{n}\sum_{i=1}^n E_i(A)(x)$ is also a decision evaluation function in [0, 1]. There are three common properties in these functions, namely regularity, nondecreasing property and complement-preserving property. This is one of our motivations to consider the axiomatic definition on complement-preserving aggregation function. Because general aggregation functions [6] satisfy regularity and nondecreasing property, aggregation functions satisfied complement-preserving property are referred to as a complement-preserving aggregation function in this paper.

And then, through these complement-preserving aggregation functions we can establish transformation methods from multiple three-way decision spaces to a single three-way decision space. These transformation methods in partially ordered set [0, 1] include the weighted average three-way decisions, max-min average three-way decisions and median three-way decisions. These methods are generalized to bi-evaluation functions.

Our method compensates for the defect of the multi-granulation rough sets which only consider two extreme models, the optimistic rough set [29] and the pessimistic rough set [27]. This paper presents more strategies for the aggregation of multi-granulation rough sets. There are the possible applications in the aggregation of the multi-granulation rough sets, the theory of multiple three-way decisions and so on.

The rest of this paper is organized as follows. Section 2, as preliminaries, recalls the decision evaluation function axioms and three-way decision spaces based on partially ordered sets. Section 3 first introduces the axiomatic definition on complement-preserving aggregation function and then gives out methods for the transformation from multiple three-way decision spaces to a single three-way decision spaces based on the axiomatic complement-preserving aggregation function. It also gives an example to illustrate these novel methods. In Section 4, these aggregation methods are generalized to three-way decisions over two groups of multiple three-way decision spaces and a practical example on evaluation of student performance is taken in order to illustrate the thoughts of the aggregation methods over two groups of multiple three-way decision spaces. Finally, Section 5 concludes the paper.

2. Preliminaries

The basic concepts, notations and results of partially ordered sets [8], decision valuation functions [7,8,11] and three-way decision spaces [7,8,11] are briefly reviewed in this section.

In this paper $(P, \leq_P)$ is a bounded partially ordered set with an involutive negator N_P, the minimum 0_P and maximum 1_P, which is written as $(P, \leq_P, N_P, 0_P, 1_P)$ [7]. In [0, 1], operator $x^c = 1 - x$ ($x \in [0, 1]$) is applied.

Let X and Y be two universes. $Map(X, Y)$ is the family of all mappings from X *to* Y, *i.e.* $Map(X, Y) = \{f | f : X \to Y\}$. If $A \in Map(U, \{0, 1\})$, then A is a subset of U, i.e. $Map(U, \{0, 1\})$ is the power set of U, which can also be written as 2^U. If $A \in Map(U, \{0, 1, [0, 1]\})$, then A is a shadowed set of U [24-25]. If $A \in Map(U, [0, 1])$, then A is a fuzzy set of U [44], namely $Map(U, [0, 1])$ is the fuzzy power set of U. If $A \in Map(U, I^{(2)})$, then A is an interval-valued fuzzy set of U [45] and an interval-valued fuzzy set A with membership function $[A^-(x), A^+(x)]$ is also denoted as $[A^-, A^+]$. If $A \in Map(U, I_s^{(2)})$, then A is an intuitionistic fuzzy set of U [1,2].

Let $(P, \leq_P, N_P, 0_P, 1_P)$ be a bounded partially ordered set. If $A \in Map(U, P)$, then the complement of A is defined pointwise by the following formula

$$N_P(A)(x) = N_P(A(x)).$$

Then $(Map(U, P), \subseteq_P, N_P, \emptyset, U)$ is a bounded partially ordered set, where $\emptyset(x) = 0_P, \forall x \in U$ and $U(x) = 1_P, \forall x \in U$, and for $A, B \in Map(U, P)$, $A \subseteq_P B$ iff $A(x) \leq_P B(x)$, $\forall x \in U$.

Let $(P_C, \leq_{P_C}, N_{P_C}, 0_{P_C}, 1_{P_C})$ and $(P_D, \leq_{P_D}, N_{P_D}, 0_{P_D}, 1_{P_D})$ be two bounded partially ordered sets in the following. Let U be a nonempty universe, on which a decision is to make. U is called a decision universe. Similarly, let V be a nonempty universe where a condition function is defined. V is named condition universe.

Definition 2.1 [8]**.** Let U be a decision universe and V be a condition universe. Then a mapping E: $Map(V, P_C) \to Map(U, P_D)$ is called a *decision evaluation function* of U, if it satisfies the following three axioms.

(E1) Minimum element axiom

$$E(\emptyset) = \emptyset, \text{ i.e., } \quad E(\emptyset)(x) = 0_{P_D}, \forall x \in U.$$

(E2) Monotonicity axiom

$$\forall A, B \in Map(V, P_C), A \subseteq_{P_C} B \Rightarrow E(A) \subseteq_{P_D} E(B), \text{ i.e., } E(A)(x) \leq_{P_D} E(B)(x), \forall x \in U.$$

(E3) Complement axiom

$$N_{P_D}(E(A)) = E(N_{P_C}(A)), \forall A \in Map(V, P_C), \text{ i.e., } N_{P_D}(E(A))(x) = E(N_{P_C}(A))(x), \forall x \in U.$$

$E(A)$ is called a decision evaluation function of U (for $A \in Map(V, P_C)$).

Given universe U, the decision condition domain $Map(V, P_C)$, decision measurement domain P_D and decision evaluation function E, then $(U, Map(V, P_C), P_D, E)$ is called a *three-way decision space*.

In multiple three-way decision spaces, two extreme transformation methods are discussed in [7], i.e., optimistic and pessimistic three-way decisions of multiple three-way decision spaces.

Definition 2.2. Let $(U, Map(V, P_C), P_D, E_i)$ $(i = 1, 2, \ldots, n)$ be n three-way decision spaces, $A \in Map(V, P_C)$, $\alpha, \beta \in P_D$ and $0 \leq \beta < \alpha \leq 1$. Then the optimistic three-way decisions of multiple three-way decision spaces are defined as follows.