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On the use of irreducible elements for reducing multi-adjoint concept lattices ☆

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ABSTRACT

Looking for strategies to reduce the size of concept lattices is very important in formal concept analysis, when they preserve the main information of the relational database.

This paper presents several properties of the useful fuzzy-attributes, in the general fuzzy case of multi-adjoint concept lattices and provides two mechanisms in order to reduce the size of concept lattices based on irreducible elements, without losing or modifying important information. Specifically, the reduced concept lattices are sublattices of the original one. Moreover, interesting properties of these mechanisms are studied and the relationship among both and other strategies is also introduced.

1. Introduction

Real databases are usually very large and give rise to complex concept lattices, from which extracting conclusions is a really difficult task. This fact highlights the importance of obtaining new procedures which lets us reduce the size of concept lattices and preserve the most relevant information of the database. Specifically, the survey of these strategies have become a key research topic in (fuzzy) Formal Concept Analysis (FCA) [20].

In the literature, we can find different mechanisms to achieve this aim. However, most of them alter the original concepts, such as the use of hedges [2,14]. Other methodologies change the original context (granular computing [13]) or consider a restrictive setting, for instance, they do not use fuzzy subsets of objects and attributes but a crisp subset of objects and a fuzzy subset of attributes, as in [17].

With the idea of providing a general framework in which the different approaches stated above could conveniently be accommodated, a new approach of formal concept analysis in which the philosophy of the multi-adjoint framework was applied and the multi-adjoint concept lattices were introduced in [18,19]. In this frame, adjoint triples [7] are the main building blocks of a multi-adjoint concept lattice and so, are used as basic operators to carry out the calculus. The flexibility provided by these triples allows us to consider a general non-commutative and non-associative environment. Moreover, different degrees of preferences related to the set of objects and attributes can easily be established in this general concept lattice framework.

On the other hand, the fuzzy notion of attributes – fuzzy-attributes – which are fuzzy subsets of attributes, are very important in representation theory of fuzzy formal concept analysis. In [9], a characterization of the meet-irreducible elements of a concept lattice from the fuzzy-attributes was presented. This characterization will play a fundamental role in the results shown in this paper, since the irreducible elements form the basic information of a relational system and consequently, they must be considered in the reduction procedures.

This paper introduces new properties about the fuzzy-attributes and presents a mechanism in order to reduce the size of multi-adjoint concept lattices from their meet-irreducible elements and a threshold given by the user. This threshold represents the least truth-value of a fuzzy-attribute in order to be considered in the computation of the concept lattice. From this reduction, a sublattice of the original concept lattice is obtained (called irreducible α-cut concept lattice), which implies that the original concepts are not modified and the most representative knowledge is preserved. However, in order to obtain the concepts of the reduced concept lattice from the initial decomposition in meet-irreducible elements and the threshold, the original concept lattice needs to be distributive. Although this is a small restriction, since other efficient

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possibilities exist to obtain the concepts of the reduced concept lattice, we have introduced another reduction mechanism, in which all the fuzzy-attributes are considered instead of the meet-irreducible elements, and the previous restriction disappears. Finally, a comparison with related strategies is presented.

The paper is organized as follows: Section 2 recalls preliminary notions and results, together with the multi-adjoint concept lattice framework; Section 3 presents several properties about fuzzy-attributes that generate meet-irreducible elements of a concept lattice. A new reduction mechanism based on meet-irreducible elements and a given threshold, together with several properties, is introduced in Section 4. Section 5 presents another strategy to reduce the size of multi-adjoint concept lattices providing the α-cut concept lattices. Lastly, a comparison with other related works is presented in Section 6. The paper finishes with several conclusions and future challenges.

2. Preliminaries

This first section provides some necessary definitions and results in order to make the paper self-contained.

Definition 1 [10]. Given a lattice $(L, \preceq)$, such that $\wedge$, $\vee$ are the meet and the join operators, and an element $x \in L$ verifying

1. If L has a top element $\top$, then $x \neq \top$.
2. If $x = y \wedge z$, then $x = y$ or $x = z$, for all $y, z \in L$.

we call x *meet-irreducible* ($\wedge$*-irreducible*) *element* of L. Condition (2) is equivalent to

2′. If $x < y$ and $x < z$, then $x < y \wedge z$, for all $y, z \in L$.

Hence, if x is $\wedge$-irreducible, then it cannot be represented as the infimum of the strictly greatest elements. A *join-irreducible* ($\vee$*-irreducible*) *element* of L is defined dually.

Note that, in a finite lattice, each element is equal to the infimum of meet-irreducible elements and the supremum of join-irreducible elements [10].

The next definition shows when the decomposition of an element of a lattice as the infimum of meet-irreducible elements is irredundant.

Definition 2 [5]. Given a lattice $(L, \preceq)$ and an element $x \in L$, if there are $\wedge$-irreducible elements $y_1, y_2, \dots, y_n$, such that $x = y_1 \wedge y_2 \wedge \cdots \wedge y_n$, then we say that x has a *finite* $\wedge$*-decomposition*. Moreover, if for each $i \in \{1, \dots, n\}$, $x \neq y_1 \wedge \cdots y_{i-1} \wedge y_{i+1} \wedge \cdots \wedge y_n$, then the decomposition is called *irredundant*, and we say that x is an *irredundant finite* $\wedge$*-decompostion.*

In the following, we recall several notions and properties related to lattice theory, which will be used later.

Definition 3 [10]. Let $(L, \preceq)$ be a lattice and $\varnothing \neq M \subseteq L$. Then $(M, \preceq)$ is a *sublattice* of $(L, \preceq)$, if for each $a, b \in M$ we have that $a \vee b \in M$ and $a \wedge b \in M$.

The following result shows when a semilattice is a complete lattice.

Lemma 4 [10]. *A complete upper (lower) semilattice* $(L, \preceq)$ *with a minimum element (maximum element) is a complete lattice.*

Next, the well-known notion of distributive lattice will be recalled.

Definition 5 [10]. A lattice $(L, \preceq)$ is called *distributive* if, for all $x, y, z \in L$, satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Observe that the above condition is equivalent to its dual: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for all $x, y, z \in L$.

The theorem below characterizes non-distributive lattices from the lattices M_3 and N_5 (see Fig. 1).

Theorem 6 [10]. *A lattice* $(L, \preceq)$ *is non-distributive if and only if* M_3 *or* N_5 *is a sublattice of* $(L, \preceq)$.

The next result shows the uniqueness of the irredundant finite $\wedge$-decompositions in a distributive lattice.

Lemma 7 [5]. *In a distributive lattice, if an element has an irredundant finite* $\wedge$*-decomposition then this decomposition is unique.*

Another definition which will play a fundamental role in one of the main theorems of this paper will be the notion of ascending chain condition.

Definition 8 [10]. Let $(P, \leqslant)$ be an ordered set. We say that P satisfies the *ascending chain condition* (ACC), if given any sequence $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$ of elements of P, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \cdots$ The dual of the ascending chain condition is the *descending chain condition* (DCC).

In a lattice, which satisfies the ascending chain condition, such as a finite lattice, each element of the lattice can be expressed from all $\wedge$-irreducible elements greater or equal to it, as the following proposition shows.

Proposition 9 [10]. *Let* $(L, \preceq)$ *be a lattice which satisfies the ascending chain condition and* $M(L)$ *the set of meet-irreducible elements of the lattice. Then, the following statement holds for all* $a \in L$:

$$a = \bigwedge \{x \in M(L) \mid a \preceq x\}$$

Once we have recalled the last proposition, the next result straightforwardly arises.

Corollary 10. *Given a lattice* $(L, \preceq)$ *satisfying the ascending chain condition, the elements* $x, z \in L$, *with* $x \preceq z$, *and the irredundant finite* $\wedge$*-decomposition of* x, $x = y_1 \wedge y_2 \wedge \cdots \wedge y_n$. *Then, there exists a subset* $K \subseteq \{1, \dots, n\}$, *such that* $z = \bigwedge_{k \in K} y_k$.

Reducing the size of multi-adjoint concept lattices is the main goal of this paper. For that, a general fuzzy setting which provides great flexibility will be considered, that is, the multi-adjoint concept lattice framework. Next, we recall this fuzzy concept lattice, which was introduced in [19].

Adjoint triples, which generalize triangular norms (t-norms) and their residuated implications [12], are considered in order to define the concept-forming operators in the multi-adjoint concept lattice framework.

Definition 11 [7]. Let $(P_1, \leqslant_1), (P_2, \leqslant_2), (P_3, \leqslant_3)$ be posets and $\& : P_1 \times P_2 \to P_3$, $\swarrow : P_3 \times P_2 \to P_1$, $\nwarrow : P_3 \times P_1 \to P_2$ be mappings, then $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to P_1, P_2, P_3 if:

$$x \leqslant_1 z \swarrow y \quad \text{iff} \quad x \& y \leqslant_3 z \quad \text{iff} \quad y \leqslant_2 z \nwarrow x \tag{1}$$

where $x \in P_1$, $y \in P_2$ and $z \in P_3$. This condition is also called *adjoint property*.

In the concept lattice environment, we need to consider that $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ are complete lattices.