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A novel cross-entropy and entropy measures of IFSs and their applications

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ABSTRACT

In this paper we discussed novel cross-entropy and entropy models on intuitionism and fuzziness of intuitionistic fuzzy sets (IFSs). In order to measure the discrimination uncertain information, cross-entropy and symmetric cross-entropy are defined based on intuitionistic factor and fuzzy factor. In particular, the constructive principle of entropy is refined; relationship between cross-entropy and entropy is investigated, so we establish a novel intuitionistic fuzzy entropy formula that simultaneously takes into account intuitionistic entropy and fuzzy entropy. We also propose concepts of marginal rate of substitution (MRS) and elasticity of substitution (ES) to describe substitution effect between intuitionism and fuzziness. Moreover, we extend the cross-entropy and entropy models to general version, which could be adjusted by different parameters, and comparative analysis with other existed cross-entropy and entropy measures are presented. Finally, we give applications of pattern recognition and decision making to demonstrate the efficiency of the cross-entropy and entropy models.

1. Introduction

In order to describe and depict the uncertainty of real word, Zadeh [1] initiated the concept of fuzzy set (FS) in 1965, which could well reflect the fuzziness nature through the use of membership degree. In 1986, Atanassov [2,3] extended FS to intuitionistic fuzzy set (IFS) by introducing non-membership degree and hesitancy degree, which make IFS more powerful and flexible in dealing with complexity and uncertainty than FS. In uncertain theory, entropy is a very important tool for measuring uncertain information, Zadeh [4] first presented fuzzy entropy to measure fuzziness by probability methods. Luca and Terini [5] discussed the constructive axioms of fuzzy entropy. Burillo and Bustince [6,7] defined intuitionistic fuzzy entropy for IFS and extended this concept to interval-valued version. Szmidt and Kacprzk [8] improved the axioms of [5] and proposed another intuitionistic fuzzy entropy based new axioms. However, there are the same disadvantages for the entropy measure in [6–8] which cannot be expressed in the equilibrium state of supportability and opposability when neutral evidences are indicated in the hesitancy degree, so Hung and Yang [9] and Wang and Lei [10] improved entropy formula and its constructive principle, but they still ignored effects inducing by changes of hesitancy degree when membership degree equal to non-membership degree, so they cannot distinguish the uncertain information in IFS which is different from FS, Pal et al. [11] pointed out that existing uncertain measures cannot capture all fuzziness and lack of knowledge and proposed a generating family of measures. In fact, entropy, distance and similarity are all basic measures in fuzzy theory, there exist many relations among them, Fan and Xie [12] studied relations between distance measure and entropy and introduce a fuzzy entropy by a divergence measure. Li et al. [13] considered relationship between similarity measure and intuitionistic fuzzy entropy, Zhang [14] presented a set of entropies for IFSs based on distance measures and intuitionistic index. Since interval-valued intuitionistic fuzzy set (IVFS) is a natural generalization of IFS, so Zeng and Li [15], Zhang et al. [16] and Wei et al. [17] analysed difference and relationship among distance, similarity and entropy in IVIFS.

In order to measure discrimination information for FSs and IFSs, Shang and Jiang [18] defined fuzzy cross-entropy between two FSs. Vlachos and Sergiadis [19] introduced the concept intuitionistic fuzzy cross-entropy, discussed relations between cross-entropy and entropy, and applied in pattern recognition, medical diagnosis and image segmentation. Since then, entropy and cross-entropy have been applied in many fields, such as portfolio selection [20], multi-criteria decision-making [21–23], and group decision-making methods [24–27].

Previous works have specialized fuzziness measures of IFSs, however, intuitionism and hesitation are also important factors. So the

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main purpose of this paper is to construct a new cross-entropy to measure discrimination uncertain information based on intuitionism and fuzziness, and refine the constructive principle of entropy for IFSs, and induce a new entropy measure with relation between cross-entropy and entropy. Meanwhile, we will give a decomposition that contains intuitionistic entropy and fuzzy entropy, and the substitution relation between them could be described by marginal rate of substitution (MRS) and elasticity of substitution (ES). Then, two parameters in cross-entropy and entropy are introduced, different effects of intuitionism and fuzziness could be presented by adjusting values of parameters. Finally, we also make comparisons with other existed cross-entropy and entropy formulas and apply cross-entropy and entropy measures proposed in this paper in pattern recognition and decision-making to demonstrate their efficiency. The rest of the paper is organized as follows. In Section 2, IFS and its operations are introduced. Section 3 proposes a new cross-entropy and entropy measure for IFS, and discusses their relations and decomposition of intuitionistic fuzzy entropy. Section 4 discusses a general model by introducing parameters in cross-entropy, and makes some comparative examples. We apply the proposed models in pattern recognition and decision-making in Section 5. Our conclusions are presented in Section 6.

2. Preliminaries

In this section, some basic concepts are illustrated, which will be needed in the following analysis.

Definition 2.1 (*IFS [2]*). Let U be an non-empty set, called the universe of discourse, an intuitionistic fuzzy set A defined on U may be given as $A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle : x\in U\}$, where $\mu_A,\gamma_A:U\to[0,1]$, and satisfy $0\leqslant\mu_A(x)+\gamma_A(x)\leqslant 1$ for all $x\in U$, the functions $\mu_A(x)$ and $\gamma_A(x)$ are called membership degree and non-membership degree to A, respectively. Denote $\pi_A(x)=1-\mu_A(x)-\gamma_A(x)$ for all $x\in U$, called hesitancy degree to A, we known that it is also a mapping from U to $[0,1]$, and satisfy $\pi_A(x)\in[0,1]$.

In fact, when $\pi_A(x)=0$ for all $x\in U$, an IFS is degenerated to a fuzzy set, so every fuzzy set could be viewed as a special case of IFS. For convenience, we denote all IFSs on U by $IFS(U)$. Similarly, let $FS(U)$ be the set of all fuzzy sets on U. Some basic operations of IFS are introduced in the following definition.

Definition 2.2 (3). Let $A, B\in IFS(U)$,

(1) Inclusion relation
$$A\subseteq B \Longleftrightarrow \mu_A(x)\leqslant\mu_B(x),\gamma_A(x)\geqslant\gamma_B(x),\forall x\in U.$$
(2) Equivalent relation
$$A=B \Longleftrightarrow \mu_A(x)=\mu_B(x),\gamma_A(x)=\gamma_B(x),\forall x\in U.$$
(3) The complement of A is defined as $A^C=\{\langle x,\gamma_A(x),\mu_A(x)\rangle : x\in U\}$.

3. Uncertain information measurement of IFSs

In this section, we only discuss the case where the universe has finite objects, i.e., $U=\{x_1,x_2,\ldots,x_m\}$. In fuzzy set theory, in order to measure the degree of discrimination between two fuzzy sets, Shang and Jiang [18] defined the fuzzy cross-entropy.

Definition 3.1 (18). Let $A, B\in FS(U)$, then the fuzzy cross-entropy of A from B is defined as

$$E(A,B)=\sum_{i=1}^{m}\left(\mu_A(x_i)ln\frac{\mu_A(x_i)}{\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i))}+(1-\mu_A(x_i))ln\frac{1-\mu_A(x_i)}{1-\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i))}\right). \quad (1)$$

Obviously, $E(A,B)$ is not symmetric with respect to its arguments, so based on this concept, Shang and Jiang [18] proposed a symmetric cross-entropy, given by $D_{FS}(A,B)=E(A,B)+E(B,A)$. Moreover, according to Shannon's inequality,we can prove that $D_{FS}(A,B)$ satisfy $0\leqslant D_{FS}(A,B)\leqslant 2mln2$ and $D_{FS}(A,B)=0$ iff (if and only if) $A=B$.

Similarly, Vlachos and Sergiadis [19] extended the fuzzy cross-entropy to intuitionistic fuzzy situation, derived a cross-entropy measure for IFSs by exploiting both the membership and the non-membership function.

Definition 3.2 (19). Let $A, B\in IFS(U)$, then the intuitionistic fuzzy cross-entropy of A against B is defined as

$$I(A,B)=\sum_{i=1}^{m}\left(\mu_A(x_i)ln\frac{\mu_A(x_i)}{\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i))}+\gamma_A(x_i)ln\frac{\gamma_A(x_i)}{\frac{1}{2}(\gamma_A(x_i)+\gamma_B(x_i))}\right). \quad (2)$$

In the same way, the symmetric intuitionistic fuzzy cross-entropy is given by $D_{IFS}(A,B)=I(A,B)+I(B,A)$. It can be easily verified that $0\leqslant D_{IFS}(A,B)\leqslant 2mln2$ and $D_{FS}(A,B)=0$ iff $A=B$, too.

3.1. Discrimination uncertain information of IFSs

Both fuzzy cross-entropy and intuitionistic fuzzy entropy describe the discrimination information of fuzzy sets or IFSs. For an arbitrary fuzzy set or IFS, discrimination information should include two parts, one is determinate information, the other is uncertain information. Furthermore, uncertain information of fuzzy sets is formed by fuzziness, which is determined by the closeness of membership and non-membership. Regarding information-driven measure used for comparison between IFS sets, the uncertain information should be described by intuitionism and fuzziness. Two definitions of fuzzy cross-entropy have recently been proposed by Shang [18] and Vlacho [19], the first one focused on the fuzziness, the second one exploited the information carried out by both membership and the non-membership function. In this section, in order to reveal the connection between the notions of entropies for IFSs in terms of fuzziness and intuitionism, we will introduce the new concept of cross-entropy which combined two factors (π_A,Δ_A).

Firstly, we suppose the universe U only has one object, i.e., $U=\{x\}$. and for an IFS $A(\in IFS(U))$, denote $\Delta_A(x)=|\mu_A(x)-\gamma_A(x)|$, in fact, the physical meaning of $\Delta_A(x)$ could be showed in a voting model such as the following example:

Example 3.1.1. Assume there are 10 people will vote on an event, where 5 people approve, 3 people against and 2 people abstain, we could show the event by intuitionistic fuzzy numbers: $A=\{\mu_A(x),\gamma_A(x)\}$, where $\mu_A(x)=3/10=0.3$ indicates level of support, $\gamma_A(x)=5/10=0.5$ means degree of opposition, naturally, $\pi_A(x)=2/10=0.2$ shows degree of hesitancy. Obviously, the power of support and opposition are different, so we denote $\Delta_A(x)=|\mu_A(x)-\gamma_A(x)|=|0.5-0.3|=0.2$ to describe balance of power between support and opposition.

Now we define intuitionistic fuzzy cross-entropy based on uncertain information as follows.

Definition 3.1.1. Let $A, B\in IFS(U)$, then the intuitionistic fuzzy cross-entropy of A against B based on uncertain information is defined as

$$CE(A,B)=\pi_A(x)ln\frac{\pi_A(x)}{\frac{1}{2}(\pi_A(x)+\pi_B(x))}+\Delta_A(x)ln\frac{\Delta_A(x)}{\frac{1}{2}(\Delta_A(x)+\Delta_B(x))}. \quad (3)$$