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Atanassov's intuitionistic fuzzy probability and Markov chains

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ABSTRACT

Fuzzy probabilities are an extension of the concept of probabilities with application in several practical problems. The former are probabilities represented through fuzzy numbers, to indicate the uncertainty in the value assigned to a probability. Moreover, Krassimir Atanassov in 1983 added an extra degree of uncertainty to classic fuzzy sets for modeling the hesitation and uncertainty about the degree of membership. This new theory of fuzzy sets is nowadays known as Atanassov intuitionistic fuzzy set theory.

This work will extend the notion of fuzzy probabilities by representing probabilities through the Atanassov intuitionistic fuzzy numbers instead of fuzzy numbers.

1. Introduction

After the introduction of the concept of fuzzy sets by Lotfi Zadeh in [69], several surveys were conducted on possible extensions of the concept of fuzzy set. Among these extensions, one that has called the attention of much research in recent decades is the Atanassov intuitionistic fuzzy set (AIFS) theory, introduced by Krassimir Atanassov in 1983 [2]. The word "intuitionistic" is used here in a "broad" sense to refer to the fact that the law of the excluded middle on the element level is denied (since $\mu_A(x) + \nu_A(x) < 1$ is possible) [22].[1] This is mainly due to the fact that AIFS is consistent with human behavior. AIFS add an extra degree to the fuzzy sets in order to model the hesitation and uncertainty about the degree of membership. In fuzzy set theory the hesitation degree (or degree of non-membership) of an element of the universe is implicitly defined as one minus the degree of membership, and therefore is fixed. In the AIFS theory the degree of hesitation, is somehow independent.

Since the AIFS theory is a generalization of the fuzzy set theory, it is natural to expect that most of the concepts and intrinsic properties of the fuzzy set theory have a counterpart in the AIFS theory. One of these, which were generalized to AIFS, are the notions of fuzzy negations, *t*-norms and *t*-conorms [24–26].

On the other hand, probability theory, is an old uncertainty method which is appropriate to deal with another kind of uncertainty. However, since probabilities consider an absolute knowledge, and in many real situations these knowledge are partially known or uncertain, there were several ways to extend the notion of probabilities, to deal with such situations. Among them we highlight the interval [16,39,66] and fuzzy. There are in the literature several different approaches of fuzzy probabilities (see for example, [11,30,53,67,70]), we highlight [11]. For James Buckley, the probabilities of events, in practice, should be known exactly, however, many times these values are estimated or provided by experts, and therefore are of vague nature. He modeled this vagueness using fuzzy numbers. This approach have been applied in several subjects (see [11,12]).

This article introduces a generalization of the concept of fuzzy probabilities representing probabilities as in [11] by using an original notion of Atanassov intuitionistic fuzzy numbers instead of usual fuzzy numbers. Thus, we give an original generalization, in the context of AIFS, to the fuzzy probability approach of James Buckley in [11] and so, we introduce a theory to deal with probabilities in a framework where it does not only model uncertainty in the probability of some events but also model the hesitation which is naturally present in the uncertainty.

In Section 2 are given the basic notions of Atanassov intuitionistic fuzzy sets and Atanassov intuitionistic fuzzy numbers that are fundamental for the paper. Note that this notion of Atanassov

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[1] Intuitionistic fuzzy logic in a "narrow" sense has been proposed by Takeuti and Titani [64].

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intuitionistic fuzzy numbers is new. In Section 3, are provided the basic definitions and results on Atanassov intuitionistic fuzzy probabilities. In Section 4 is introduced an Atanassov intuitionistic fuzzy extension of (fuzzy) conditional probability and it is proved that this extension satisfies analogous properties than (crisp) conditional probabilities. Notice that all results in this last two sections are new. In Section 5 we introduce the notion of Atanassov intuitionistic fuzzy Markov chains and prove some of its basic properties. In Section 6, we present a case of study where the use of Atanassov's intuitionistic fuzzy probability contribute to calculate the probability taking into account uncertainty as well as hesitation which are inherent to the problem. The last section is devoted to the final remarks and future works.

We assume that the reader is familiar with fuzzy logic, probability and their fuzzy versions. For more details on these theories see [11] and for Atanassov intuitionistic fuzzy set theory see [4,14,51,54].

2. Atanassov intuitionistic fuzzy numbers

An Atanassov intuitionistic fuzzy set (AIFS) A in a universe X is a set

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$$

where μ_A, ν_A:$X \to [0,1]$ satisfy the condition $\mu_A(x) + \nu_A(x) \leqslant 1$. $\mu_A(X)$ and $\nu_A(x)$ denote the membership degree and the non-membership degree of the element x in the set A, respectively.

Glad Deschrijver and Etiene Kerre in [27] give an alternative approach to AIFS, they proved that AIFS can also be seen as a L^*-valued fuzzy set in the sense of Joseph Goguen [33] for consider the complete lattice $\langle L^*, \leqslant_{L^*}\rangle$ where

$$L^* = \{(x,y) \in [0,1] \times [0,1] | x + y \leqslant 1\}$$

and

$$(x_1,x_2) \leqslant_{L^*} (y_1,y_2) \text{if and only if } x_1 \leqslant y_1 \text{ and } x_2 \geqslant y_2.$$

Note that $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Thus, AIFS are nothing more than L^*-valued fuzzy sets.

Given a subset $A \subseteq L^*$ its supremum and infimum, with respect to $\leqslant_{L^*}$, could be obtained as follows:

$$\sup A = (\sup\{\pi_1(x) | x \in A\}, \inf\{\pi_2(x) | x \in A\})$$
$$\inf A = (\inf\{\pi_1(x) | x \in A\}, \sup\{\pi_2(x) | x \in A\})$$

where $\pi_1(x_1,x_2) = x_1$ and $\pi_2(x_1,x_2) = x_2$ are the natural projections.

Given $(\alpha,\beta) \in L^+$ where $L^+ = L^* - \{0_{L^*}\}$ and an AIFS A of universe X, the (α,β)-cut of A is the set

$$A_{(\alpha,\beta)} = \{x \in X | \mu_A(x) \geqslant \alpha \text{ and } \nu_A(x) \leqslant \beta\}. \quad (1)$$

Notice that $A_{(\alpha,\beta)} = A_\alpha \cap A^\beta$ where $A_\alpha = \{x \in X | \mu_A(x) \geqslant \alpha\}$ and $A^\beta = \{x \in X | \nu_A(x) \leqslant \beta\}$.

Notice that every AIFS can be recovered from its (α,β)-cuts. In fact

$$(\mu_A(x), \nu_A(x)) = \sup\{(\alpha,\beta) \in L^* | x \in A_{(\alpha,\beta)}\}$$

where the supremum is with respect to $\leqslant_{L^*}$.

Definition 2.1. Let A be an AIFS with universe $\mathbb{R}$. A is **connected** if every (α,β)-cut of A is connected, i.e., interval of real numbers. A is **normalized** if there exists $x \in \mathbb{R}$ such that $(\mu_A(x), \nu_A(x)) = 1_{L^*}$.

Thus, for a connected AIFS, its (α,β)-cuts are closed intervals of real numbers, and so it is possible to represent them via their end points, which can be obtained as follows:

$$l(A_{(\alpha,\beta)}) = \max\left(\min \mu_A^{-1}(\alpha), \min \nu_A^{-1}(\beta)\right) \text{ and} \quad (2)$$
$$r(A_{(\alpha,\beta)}) = \min\left(\max \mu_A^{-1}(\alpha), \max \nu_A^{-1}(\beta)\right). \quad (3)$$

That is, $A_{(\alpha,\beta)} = [l(A_{(\alpha,\beta)}),r(A_{(\alpha,\beta)})]$. Since A is convex, $|\nu_A^{-1}(\beta)|, |\mu_A^{-1}(\alpha)| \leqslant 2$.

Definition 2.2. An AIFS A with the real universe is an **Atanassov intuitionistic fuzzy number** (AIFN) if it is connected, normalized and $\mu_A$and ν_A are piecewise continuous.

There are several definitions for AIFN, as for example [7,13,35,37,44,46,47,49]. In particular, Atanassov in [5] discusses some ideas for different types of AIFN. Nevertheless, our definition is different from these proposals because is the unique based on (α,β)-cuts. Moreover, our definition is also non equivalent with some of these definitions. For example, in [7,37] require that μ_A be a fuzzy number (in the sense of [17]) with a bounded support and in our case μ_A is also a fuzzy number, but in the sense of page 26 in [29] which require that μ_A be piecewise continuous instead of just continuous. In [47] no requirements on continuity is made. For us and for [7], ν_A^c (the complement of ν_A, i.e. $\nu_A^c(x) = 1 - \nu_A(x)$) should also be a fuzzy number, related in some sense with μ_A, but for [37] ν_A^c does not need to be a fuzzy number. Furthermore, in [44] it is required that μ_A should be upper semi continuous and ν_A should be lower semi continuous, whereas we require that both should be pairwise continuous.

An important subclass of AIFN are those in which μ_A, as well as ν_A^c, have a triangular shape, that is, are triangular fuzzy numbers in the usual sense (TFN), that is why they will be called triangular Atanassov intuitionistic fuzzy numbers (TAIFN). Thus, as can be seen in Fig. 1, these AIFN are completely determined by the values that characterize the TAIFN, μ_A and ν_A^c.

Thus, a TAIFN like Fig. 1, will be denoted by $(a,b/c/d,e)$. One advantage of TAIFN with respect to any AIFS, is that its (α,β)-cuts can be easily determined as follows:

$$(a,b/c/d,e)_{(\alpha,\beta)} = [\max(a + (c-a)\alpha, b + (c-b)\beta), \times \min(e + (e-c)\alpha, d + (d-c)\beta)] \quad (4)$$

There are different ways of define the arithmetic operations on fuzzy numbers, see for example [6,28,47,58]. Particularly, in [6,58] were introduced some types of "division" for AIFN. Here we will introduce arithmetic operations based on (α,β)-cuts as made in [47], but note that the notions of (α,β)-cuts are differents, once that for us each (α,β)-cuts determines an interval, whereas for [47] is a pair of intervals. Let A and B be two AIFN. Then define the addition, subtraction, multiplication and division of A with B from the corresponding interval arithmetic operations on their (α,β)-cuts. Let $(\alpha,\beta) \in L^+$,

$$(A+B)_{(\alpha,\beta)} = \{x + y | x \in A_{(\alpha,\beta)} \text{and } y \in B_{(\alpha,\beta)}\} = [l(A_{(\alpha,\beta)}) + l(B_{(\alpha,\beta)}), r(A_{(\alpha,\beta)}) + r(B_{(\alpha,\beta)})]$$

$$(A-B)_{(\alpha,\beta)} = \{x - y | x \in A_{(\alpha,\beta)} \text{and} y \in B_{(\alpha,\beta)}\} = [l(A_{(\alpha,\beta)}) - r(B_{(\alpha,\beta)}), r(A_{(\alpha,\beta)}) - l(B_{(\alpha,\beta)})]$$

Table 1
Current table of the South American qualifying zone for the 2014 FIFA World Cup.

Position	Country	Plays	Win	Tie	Lose	Points
1	Argentine	9	6	2	1	20
2	Equator	9	5	2	2	17
3	Colombia	8	5	1	2	16
4	Venezuela	9	3	3	3	12
5	Uruguay	9	3	3	3	12
6	Chile	9	4	0	5	12
7	Bolivia	9	2	2	5	8
8	Peru	9	2	2	5	8
9	Paraguay	9	2	1	6	7