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Random sampling in computational algebra: Helly numbers and violator spaces

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ABSTRACT

This paper transfers a randomized algorithm, originally used in geometric optimization, to computational problems in commutative algebra. We show that Clarkson's sampling algorithm can be applied to two problems in computational algebra: solving large-scale polynomial systems and finding small generating sets of graded ideals. The cornerstone of our work is showing that the theory of violator spaces of Gärtner et al. applies to polynomial ideal problems. To show this, one utilizes a Helly-type result for algebraic varieties. The resulting algorithms have expected runtime linear in the number of input polynomials, making the ideas interesting for handling systems with very large numbers of polynomials, but whose rank in the vector space of polynomials is small (e.g., when the number of variables and degree is constant).

1. Introduction

Many computer algebra systems offer excellent algorithms for manipulation of polynomials. But despite great success in the field, many algebraic problems have bad worst-case complexity. For example, Buchberger's (Buchberger, 1965, 2006; Cox et al., 2007) groundbreaking algorithm, key to

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symbolic computational algebra today, computes a Gröbner basis of any ideal, but it has a worst-case runtime that is doubly exponential in the number of variables (Dube, 1990). This presents the following problem: *What should one do about computations whose input is a very large, overdetermined system of polynomials? In this paper, we propose to use randomized sampling algorithms to ease the computational cost in such cases.*

One can argue that much of the success in computation with polynomials (of non-trivial size) often relies heavily on finding specialized structures. Examples include Faugère's et al. fast computation of Gröbner bases of zero-dimensional ideals (Faugére et al., 2013, 1993, 2014; Gao, 2009; Lakshman, 1990), specialized software for computing generating sets of toric ideals `4ti2` (team 4ti2), several packages in `Macaulay2` (Grayson and Stillman) built specifically to handle monomial ideals, and the study of sparse systems of polynomials (i.e., systems with fixed support sets of monomials) and the associated homotopy methods (Sturmfels, 1991). A more recent example of the need to find good structures is in Cifuentes and Parrilo (2015), where the authors began exploiting chordal graph structure in computational commutative algebra, and in particular, for solving polynomial systems. Our paper exploits combinatorial structure implicit in the input polynomials, but this time akin to Helly-type results from convex discrete geometry (Matoušek, 2002).

At the same time, significant improvements in efficiency have been obtained by algorithms that involve randomization, rather than deterministic ones (e.g. Berlekamp, 1970; Solovay and Strasse, 1977); it is also widely recognized that there exist hard problems for which pathological examples requiring exponential runtimes occur only rarely, implying an obvious advantage of considering average behavior analysis of many algorithms. For example, some forms of the simplex method for solving linear programming problems have worst-case complexity that is exponential, yet Spielman and Teng (2009) have recently shown that in the *smoothed analysis of algorithms* sense, the simplex method is a rather robust and fast algorithm. Smoothed analysis combines the worst-case and average-case algorithmic analyses by measuring the expected performance of algorithms under slight random perturbations of worst-case inputs. Of course, probabilistic analysis, and smoothed analysis in particular, has been used in computational algebraic geometry for some time now, see e.g., the elegant work in Beltrán and Pardo (2008, 2009), Bürgisser and Cucker (2011). The aim of this paper is to import a randomized sampling framework from geometric optimization to applied computational algebra, and demonstrate its usefulness on two problems.

Our contributions

We apply the theory of violator spaces (Gärtner et al., 2008) to polynomial ideals and adapt Clarkson's sampling algorithms (Clarkson, 1995) to provide efficient randomized algorithms for the following concrete problems:

(1) *solving large (overdetermined) systems of multivariate polynomials equations,*
(2) *finding small, possibly minimal, generating sets of homogeneous ideals.*

Our method is based on using the notion of a *violator space*. Violator spaces were introduced in 2008 by Gärtner, Matoušek, Rüst, and Škovroň (Gärtner et al., 2008) in a different context. Our approach allows us to adapt Clarkson's sampling techniques (Clarkson, 1995) for computation with polynomials. Clarkson-style algorithms rely on computing with small-size subsystems, embedded in an iterative biased sampling scheme. In the end, the local information is used to make a global decision about the entire system. The expected runtime is linear in the number of input elements, which is the number of polynomials in our case (see Brise and Gärtner, 2009 for a more recent simplified version of Clarkson's algorithm for violator spaces). Violator spaces naturally appear in problems that have a natural linearization and a sampling size given by a combinatorial *Helly number* of the problem. While violator spaces and Clarkson's algorithm have already a huge range of applications, to our knowledge, this is the first time such sampling algorithms are being used in computational algebraic geometry. For an intuitive reformulation of Helly's theorem for algebraic geometers, see Example 3. Main ingredients of violator spaces are illustrated through Examples 4, 7 and 11. A typical